Week 5. Why Classical Solutions Fail

Part II. Hamilton–Jacobi Equations and Viscosity Solutions

Primary reference

- H. V. Tran, *Hamilton–Jacobi Equations: Theory and Applications*, §1.1 and §1.2.

In the previous weeks, we derived Hamilton–Jacobi–Bellman equations from deterministic optimal control and the dynamic programming principle. At that stage, the PDE appeared naturally as the equation satisfied by the value function. However, that derivation alone does not answer a fundamental analytic question: **in what sense should the PDE be solved when the value function is not smooth?**

This lecture begins the second part of the course. Our goal is to explain why the classical notion of solution is too restrictive for Hamilton–Jacobi equations, and why a more robust notion of generalized solution is necessary. The key message is that Hamilton–Jacobi equations are often meaningful **before** they are smooth.

1. Learning Objectives

By the end of this lecture, students should be able to:

1. explain why Hamilton–Jacobi equations arise in several different applications;
2. identify the limitations of the classical C^1 solution concept;
3. analyze the one-dimensional eikonal boundary value problem

$$|u'(x)| = 1 \quad \text{in } (-1,1), \qquad u(-1) = u(1) = 0;$$

4. prove that this problem has no classical solution;
5. construct infinitely many almost everywhere solutions;
6. understand why a **selection principle** is needed;
7. see viscosity solutions as the correct answer to this selection problem.

2. From Optimal Control Back to PDE

In Weeks 2–4, we studied deterministic control systems and the value function associated with an optimization problem. Under the dynamic programming principle, the value function formally satisfies a Hamilton–Jacobi–Bellman equation of the form

$$\lambda u(x) + H(x, Du(x)) = 0.$$

This perspective is conceptually powerful: it explains where the PDE comes from. But it also reveals a serious difficulty. Even when the control problem is well posed, the value function need not be differentiable. Indeed, nonsmoothness is typical rather than exceptional.

Therefore, once the PDE has been derived, the next question is unavoidable:

> What is the correct notion of solution for a Hamilton–Jacobi equation when classical differentiability fails?

This question motivates the theory of viscosity solutions.

3. Why Hamilton–Jacobi Equations Arise

Hamilton–Jacobi equations appear in many different settings. The importance of this observation is structural: the same analytic difficulties occur across mechanics, control, and geometry.

3.1 Classical Mechanics

A standard Hamiltonian from mechanics is

$$H(x,p) = \frac{1}{2}|p|^2 + V(x),$$

where x is position and p is the momentum variable. The associated Hamilton–Jacobi equation describes an action function whose gradient encodes the momentum field.

Even in this classical setting, smooth solutions may fail to exist globally because characteristics can intersect.

3.2 Front Propagation

Suppose a moving front evolves with normal speed $a(x)$. In the level set formulation, the motion is described by

$$u_t + a(x)|Du| = 0.$$

The factor $|Du|$ reflects the fact that only the normal direction matters in the propagation law.

3.3 G-equation

In combustion models, one also includes transport by a background velocity field $W(x)$. This leads to

$$u_t + |Du| + W(x) \cdot Du = 0.$$

This equation combines normal propagation with advection.

3.4 Level Set Mean Curvature Flow

A geometric evolution law of a different type is, formally at points where $Du \neq 0$,

$$u_t = |Du| \operatorname{div}\left(\frac{Du}{|Du|}\right).$$

This is no longer first-order, but it belongs to the same broader framework in which singularities and generalized solutions play a central role.

3.5 Main Lesson

These examples come from different applications, but they lead to the same conclusion:

- the PDE is natural;
- the natural solution may fail to be smooth;
- therefore classical solvability is not the right framework.

4. Classical Solutions and Their Limitation

We begin with the standard notion.

Definition 4.1 (Classical Solution)

Let $\Omega \subset \mathbb{R}^n$ be open, and let

$$F : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$$

be continuous. A function u is a **classical solution** of the first-order equation

$$F(x, u(x), Du(x)) = 0 \quad \text{in } \Omega$$

if $u \in C^1(\Omega)$ and the equation holds pointwise for every $x \in \Omega$.

We use $F(x, r, p)$ for the full PDE expression, where r represents the value of the unknown function and p its gradient. We reserve $H(x, p)$ for the **Hamiltonian**. For example, the discounted HJB equation

$$\lambda u + H(x, Du) = 0$$

corresponds to

$$F(x, r, p) = \lambda r + H(x, p).$$

If time is present, for example in the Hamilton–Jacobi equation

$$u_t + H(x, Du) = 0,$$

then a classical solution is a function u that is C^1 in (x, t) and satisfies the equation pointwise.

This definition is mathematically natural, but for Hamilton–Jacobi equations it is often too restrictive. The right solution may be continuous or Lipschitz, yet fail to be differentiable at key points.

5. Core Example: A One-Dimensional Eikonal Problem

We now study the most important example of this lecture:

$$|u'(x)| = 1 \quad \text{in } (-1, 1), \qquad u(-1) = u(1) = 0.$$

This problem is elementary enough to be analyzed completely, yet rich enough to show the essential failure of the classical framework.

The equation says that wherever the derivative exists, the slope must have magnitude one. Thus any candidate solution must be built from line segments of slope $+1$ or -1.

6. Nonexistence of Classical Solutions

We first show that the problem has no classical solution.

Proposition 6.1

The boundary value problem

$$|u'(x)| = 1 \quad \text{in } (-1, 1), \qquad u(-1) = u(1) = 0$$

has no classical solution.

Proof

Assume that $u \in C^1((-1, 1)) \cap C([-1, 1])$ is a classical solution. Then for every $x \in (-1, 1)$,

$$|u'(x)| = 1,$$

so

$$u'(x) \in \{-1, 1\}.$$

Since u' is continuous and $(-1, 1)$ is connected, the image of u' must also be connected. But the set $\{-1, 1\}$ is disconnected. Therefore u' must be constant.

Hence either

$$u'(x) = 1 \quad \text{for all } x \in (-1, 1),$$

or

$$u'(x) = -1 \quad \text{for all } x \in (-1, 1).$$

In the first case,

$$u(1) - u(-1) = \int_{-1}^{1} u'(x)\,dx = 2,$$

which contradicts the boundary conditions $u(-1) = u(1) = 0$.

In the second case,

$$u(1) - u(-1) = \int_{-1}^{1} u'(x)\,dx = -2,$$

which again contradicts the boundary conditions.

Therefore no classical solution exists. □

Remark 6.2

The importance of this example is that the obstruction is not technical. The PDE is simple, the boundary data are simple, and yet classical solvability fails completely.

7. Almost Everywhere Solutions

If the classical notion is too strong, a natural first attempt is to weaken the equation and ask that it hold almost everywhere.

Definition 7.1 (a.e. Solution)

A Lipschitz function $u : [-1, 1] \to \mathbb{R}$ is called an **almost everywhere solution** of

$$|u'| = 1 \quad \text{in } (-1, 1)$$

if u is differentiable almost everywhere and

$$|u'(x)| = 1$$

for almost every $x \in (-1, 1)$.

This definition allows nondifferentiable points, so it is much broader than the classical notion.

8. Infinitely Many a.e. Solutions

The problem with the a.e. notion is not existence, but nonuniqueness.

Example 5.1 (Symmetric Tent Function)

Define

$$u_1(x) = 1 - |x|.$$

Then $u_1(-1) = u_1(1) = 0$, and for almost every $x \neq 0$,

$$u_1'(x) = \begin{cases} 1, & x < 0, \\ -1, & x > 0. \end{cases}$$

Hence

$$|u_1'(x)| = 1$$

for almost every $x \in (-1, 1)$. Therefore u_1 is an a.e. solution.

Example 5.2 (Negative Tent Function)

Define

$$u_2(x) = -1 + |x|.$$

Again $u_2(-1) = u_2(1) = 0$, and $|u_2'(x)| = 1$ almost everywhere. Thus u_2 is also an a.e. solution.

Already this shows nonuniqueness.

Example 5.3 (Sawtooth-Type Solutions)

We can construct many more examples. Consider any continuous piecewise linear function on $[-1, 1]$ such that:

- it equals 0 at $x = -1$ and $x = 1$;
- each linear piece has slope either $+1$ or -1.

Then the derivative exists almost everywhere and satisfies $|u'| = 1$ almost everywhere. Hence every such function is an a.e. solution.

This gives infinitely many solutions.

A General Construction

Let $\sigma : (-1, 1) \to \{-1, 1\}$ be measurable and satisfy

$$\int_{-1}^{1} \sigma(s)\, ds = 0.$$

Define

$$u(x) = \int_{-1}^{x} \sigma(s)\, ds.$$

Then $u(-1) = 0$ and

$$u(1) = \int_{-1}^{1} \sigma(s)\, ds = 0.$$

Moreover,

$$u'(x) = \sigma(x)$$

for almost every x, so

$$|u'(x)| = 1$$

almost everywhere. Therefore u is an a.e. solution.

Since there are infinitely many such functions σ, there are infinitely many a.e. solutions.

The three profiles below all satisfy the boundary conditions and have slopes ± 1 almost everywhere. The figure makes the selection problem visible: checking the equation only where a derivative exists cannot distinguish the distance profile from its sign reversal or from oscillatory sawtooth competitors.

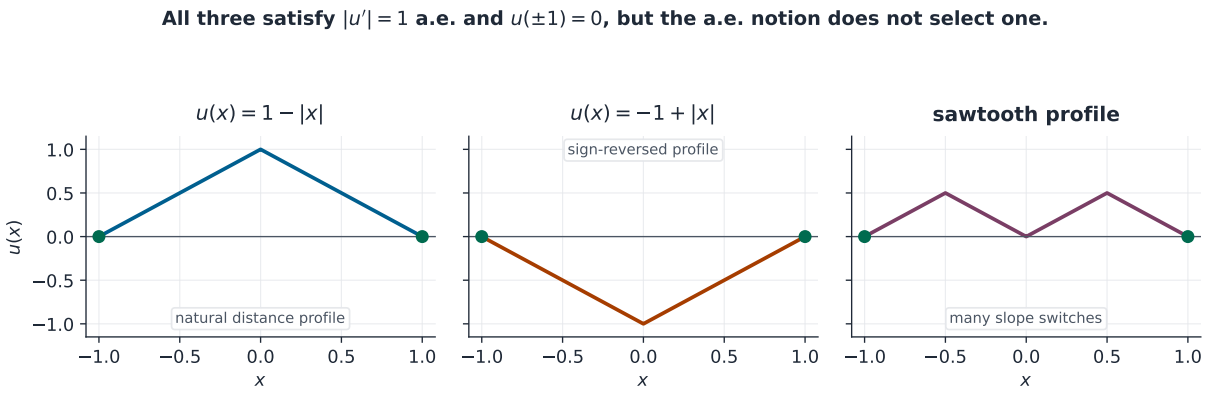


Figure 5.1. The a.e. equation $|u'| = 1$ admits many solutions with $u(\pm 1) = 0$. A useful generalized solution concept must retain the geometrically meaningful tent profile while excluding the other competitors as full viscosity solutions.

9. Symmetry and the Selection Problem

The example reveals a second structural issue: symmetry.

If u is an a.e. solution of

$$|u'| = 1,$$

then so is $-u$. In particular, the PDE and boundary condition do not distinguish between

$$u(x) = 1 - |x| \quad \text{and} \quad u(x) = -1 + |x|.$$

This is mathematically problematic because in many applications one expects a distinguished solution. For example, in distance-type or minimum-time interpretations, the natural solution should be nonnegative and should represent a meaningful geometric quantity. The equation in the naive a.e. sense does not encode that preference.

Therefore the problem is not merely to enlarge the solution class. We must enlarge it in a way that still preserves:

1. uniqueness;
2. stability;
3. compatibility with the underlying model.

This is the role of viscosity solutions.

10. A Two-Dimensional Analogue

Consider the higher-dimensional problem

$$|Du| = 1 \quad \text{in } B_1(0), \qquad u = 0 \quad \text{on } \partial B_1(0).$$

A natural candidate is

$$u(x) = 1 - |x|.$$

This function is continuous on $\overline{B_1(0)}$, vanishes on the boundary, and satisfies

$$|Du(x)| = 1$$

for every $x \neq 0$. However, u is not differentiable at the origin.

Thus even the most geometrically natural solution is not classical.

The same basic difficulty persists:

- the correct solution exists naturally;

- it is not C^1;
- a generalized framework is required.

Later in the course we will prove that the distance-to-the-boundary function is the unique viscosity solution of this problem.

11. Why This Matters for Hamilton–Jacobi Theory

The failure of classical solutions is not an isolated phenomenon. It reflects a general mechanism.

11.1 Intersecting Characteristics

Hamilton–Jacobi equations propagate information along characteristic curves. When characteristics intersect, differentiability is typically lost.

11.2 Competition Between Optimal Strategies

In optimal control, the value function arises from minimization. When several strategies are equally optimal, the value function may develop corners or kinks.

11.3 Geometric Singularities

Distance functions, arrival-time functions, and level set functions are often only Lipschitz, even when they are the correct objects from the geometric point of view.

Therefore the correct theory must do three things at once:

1. include nonsmooth but meaningful functions;
2. exclude spurious weak solutions;
3. restore well-posedness.

12. Preview of the Viscosity Framework

Week 5 is not yet the week in which we solve the problem. Instead, it clarifies exactly what the problem is.

For the one-dimensional eikonal example, we have established:

- there is **no classical solution**;
- there are **infinitely many a.e. solutions**;
- the equation has a **sign symmetry** $u \mapsto -u$;
- therefore a **selection principle** is necessary.

In the next lecture, we will introduce the idea of **vanishing viscosity** and define viscosity subsolutions and supersolutions using smooth test functions. That definition will retain the natural nonsmooth candidate $u(x) = 1 - |x|$, while excluding oscillatory competitors such as sawtooth functions as full viscosity solutions.

13. Summary

The main lesson of this lecture is the following:

> Hamilton–Jacobi equations are often meaningful before they are smooth.

If we insist on classical C^1 solutions, we may lose the natural solution entirely. If we weaken the equation too much, we lose uniqueness. Viscosity theory is designed precisely to resolve this tension.

14. Exercises

Exercise 5.1

Show that

$$u(x) = 1 - |x|$$

is an a.e. solution of

$$|u'(x)| = 1 \quad \text{in } (-1,1), \qquad u(-1) = u(1) = 0.$$

Exercise 5.2

Construct a nontrivial sawtooth-type function on $[-1,1]$ with endpoint values 0 and slopes ± 1, and verify that it is an a.e. solution.

Exercise 5.3

Explain why the proof of Proposition 6.1 depends essentially on the continuity of u'.

Exercise 5.4

Verify formally that

$$u(x) = 1 - |x|$$

satisfies $|Du| = 1$ in $B_1(0) \setminus \{0\}$.

Exercise 5.5

Write a short paragraph explaining why the existence of infinitely many a.e. solutions means that the a.e. notion is not a well-posed solution concept.

15. Concluding Remark

The one-dimensional eikonal example is simple enough to be understood completely, but deep enough to motivate an entire theory. It shows that the central issue in Hamilton–Jacobi equations is not merely solving a PDE, but identifying the **correct** notion of solution.

References

- H. V. Tran, *Hamilton–Jacobi Equations: Theory and Applications*, §1.1–§1.2.