Week 6. Vanishing Viscosity and Viscosity Solutions

Hamilton–Jacobi Equations and Optimal Control

Primary reference

- H. V. Tran, *Hamilton–Jacobi Equations: Theory and Applications*, §1.2, especially Definition 1.5.

1. Position of this lecture in the course

In this course, we do **not** follow the textbook order literally.
Instead, we first study **optimal control**, the **dynamic programming principle**, and the derivation of the **Hamilton–Jacobi / Hamilton–Jacobi–Bellman equation**, and only then introduce **viscosity solutions**.

This order is mathematically natural for our purposes. By the time we arrive at viscosity solutions, we already understand:

- where the Hamilton–Jacobi equation comes from,
- why the value function is the central object,
- and why the value function is typically **not smooth**.

Therefore, viscosity solutions should be understood as the correct weak notion of solution for Hamilton–Jacobi equations arising from optimal control.

2. Main goals of this lecture

By the end of this lecture, you should be able to:

1. explain why classical C^1 solutions are often too restrictive for Hamilton–Jacobi equations;
2. understand the **vanishing viscosity approximation**

$$u_t^\varepsilon + H(x, Du^\varepsilon) = \varepsilon \Delta u^\varepsilon;$$

3. derive formally the viscosity inequalities from the parabolic approximation;
4. state the definition of viscosity subsolutions, supersolutions, and viscosity solutions;
5. understand the flexibility of the definition;
6. verify in basic examples whether a non-smooth function is a viscosity solution.

3. Why classical solutions are not enough

Consider the Cauchy problem

$$u_t + H(x, Du) = 0 \quad \text{in } \mathbb{R}^n \times (0, T), \qquad u(x, 0) = u_0(x).$$

In optimal control, the unknown u is typically the **value function**, obtained from an infimum over admissible controls. Such a function is usually continuous, often Lipschitz, but in general **not differentiable everywhere**.

This creates a serious difficulty. The equation contains the gradient Du, but Du may fail to exist exactly at the most important points, namely points where several competing optimal behaviors meet.

Thus:

- the notion of **classical solution** is too restrictive;
- the notion of **almost everywhere solution** is too weak.

We need a notion of solution that:

- allows non-smooth functions,

- remains stable under approximation,
- and still preserves the correct PDE structure.

This is the role of **viscosity solutions**.

4. Vanishing viscosity: the basic idea

To regularize the first-order Hamilton–Jacobi equation, we add a small diffusion term:

$$u_t^\varepsilon + H(x, Du^\varepsilon) = \varepsilon \Delta u^\varepsilon, \qquad \varepsilon > 0.$$

For each fixed $\varepsilon > 0$, this is a **parabolic equation**.
Under appropriate regularity and growth assumptions, parabolic theory gives smoother approximating solutions and a maximum principle.

The idea is:

1. solve the regularized parabolic problem for each $\varepsilon > 0$;
2. obtain useful estimates independent of ε;
3. let $\varepsilon \to 0$;
4. interpret the limit as the correct solution of the original first-order problem.

This procedure is called the **vanishing viscosity method**.

5. Why the maximum principle is the key

The essential advantage of the viscous equation

$$u_t^\varepsilon + H(x, Du^\varepsilon) = \varepsilon \Delta u^\varepsilon$$

is that the diffusion term allows us to use the maximum principle.

At a local maximum of a smooth function, the Laplacian is nonpositive.
At a local minimum, the Laplacian is nonnegative.

These sign conditions are exactly what survive in the limit $\varepsilon \to 0$, and they lead to the one-sided inequalities that define viscosity subsolutions and supersolutions.

So the viscosity definition is not an arbitrary invention. It is the natural limiting structure suggested by the parabolic approximation.

6. Formal derivation of the viscosity subsolution inequality

Assume that for each $\varepsilon > 0$,

$$u_t^\varepsilon + H(x, Du^\varepsilon) = \varepsilon \Delta u^\varepsilon,$$

and suppose that

$$u^\varepsilon \to u \quad \text{locally uniformly as } \varepsilon \to 0.$$

For this formal vanishing-viscosity argument, let $\varphi \in C^{2,1}(\mathbb{R}^n \times (0,T))$, with two spatial derivatives and one time derivative, and suppose that $u - \varphi$ has a **strict local maximum** at (x_0, t_0), with

$$u(x_0, t_0) = \varphi(x_0, t_0).$$

Because $u^\varepsilon \to u$ locally uniformly, for sufficiently small ε, the function

$$u^\varepsilon - \varphi$$

has a local maximum at some point $(x_\varepsilon, t_\varepsilon)$ near (x_0, t_0), and

$$(x_\varepsilon, t_\varepsilon) \to (x_0, t_0).$$

At such a local maximum, we have:

- first derivative condition:

$$Du^\varepsilon(x_\varepsilon, t_\varepsilon) = D\varphi(x_\varepsilon, t_\varepsilon);$$

- time derivative condition:

$$u_t^\varepsilon(x_\varepsilon, t_\varepsilon) = \varphi_t(x_\varepsilon, t_\varepsilon);$$

- second derivative inequality:

$$\Delta u^\varepsilon(x_\varepsilon, t_\varepsilon) \le \Delta\varphi(x_\varepsilon, t_\varepsilon).$$

Evaluating the viscous equation at $(x_\varepsilon, t_\varepsilon)$, we obtain

$$\varphi_t(x_\varepsilon, t_\varepsilon) + H(x_\varepsilon, D\varphi(x_\varepsilon, t_\varepsilon)) = \varepsilon\Delta u^\varepsilon(x_\varepsilon, t_\varepsilon) \le \varepsilon\Delta\varphi(x_\varepsilon, t_\varepsilon).$$

Passing to the limit $\varepsilon \to 0$, we formally get

$$\varphi_t(x_0, t_0) + H(x_0, D\varphi(x_0, t_0)) \le 0.$$

This is exactly the **viscosity subsolution inequality**.
Similarly, if $u - \psi$ has a strict local minimum at (x_0, t_0), then the same argument gives

$$\psi_t(x_0, t_0) + H(x_0, D\psi(x_0, t_0)) \ge 0,$$

which is the **viscosity supersolution inequality**.

7. Definition of viscosity solutions

We now state the definition for the Cauchy problem

$$u_t + H(x, Du) = 0 \quad \text{in } \mathbb{R}^n \times (0, T), \qquad u(x, 0) = u_0(x).$$

Assume

$$u \in \mathrm{BUC}(\mathbb{R}^n \times [0, T)),$$

that is, u is bounded and uniformly continuous.
We assume $u_0 \in \mathrm{BUC}(\mathbb{R}^n)$ and that H is continuous. The definition below is the standard x-dependent extension of Tran's Definition 1.5.

Definition: viscosity subsolution

A function $u \in \mathrm{BUC}(\mathbb{R}^n \times [0, T))$ is called a **viscosity subsolution** if for every

$$\varphi \in C^1(\mathbb{R}^n \times (0, T)),$$

whenever $u - \varphi$ has a strict local maximum at $(x_0, t_0) \in \mathbb{R}^n \times (0, T)$ and

$$u(x_0, t_0) = \varphi(x_0, t_0),$$

we have

$$\varphi_t(x_0, t_0) + H(x_0, D\varphi(x_0, t_0)) \le 0,$$

and also

$$u(\cdot, 0) \le u_0.$$

Definition: viscosity supersolution

A function $u \in \mathrm{BUC}(\mathbb{R}^n \times [0,T))$ is called a **viscosity supersolution** if for every

$$\psi \in C^1(\mathbb{R}^n \times (0,T)),$$

whenever $u - \psi$ has a strict local minimum at $(x_0, t_0) \in \mathbb{R}^n \times (0,T)$ and

$$u(x_0,t_0) = \psi(x_0,t_0),$$

we have

$$\psi_t(x_0,t_0) + H(x_0, D\psi(x_0,t_0)) \geq 0,$$

and also

$$u(\cdot,0) \geq u_0.$$

Definition: viscosity solution

A function is called a **viscosity solution** if it is both a viscosity subsolution and a viscosity supersolution.

8. Remarks on the definition

8.1. Why test functions?

We do not require the unknown function u itself to be differentiable.
Instead, we use smooth test functions that touch u locally from above or below.
The PDE is then imposed on the smooth test function at the touching point.

8.2. Why inequalities rather than equality?

At non-smooth points, there may be only one-sided touching information.
Therefore, the correct formulation is one-sided:

- touching from above gives the **subsolution inequality**;
- touching from below gives the **supersolution inequality**.

8.3. Why inequalities for the initial condition?

Subsolutions satisfy

$$u(\cdot,0) \leq u_0,$$

and supersolutions satisfy

$$u(\cdot,0) \geq u_0.$$

If both hold, then

$$u(\cdot,0) = u_0.$$

9. Flexibility of the definition

The definition above is more robust than it may first appear.

9.1. Choice of test functions

Although the definition is stated using C^1 test functions, one can equivalently use:

- C^2 test functions,
- or even C^∞ test functions.

So the precise smoothness class of the test functions is not essential.

9.2. Strict local extrema vs. local extrema

The condition of **strict** local maximum/minimum can be replaced by ordinary local maximum/minimum. The two formulations are equivalent after a standard perturbation argument.

9.3. Equality at the touching point

The condition

$$u(x_0, t_0) = \varphi(x_0, t_0)$$

is essentially automatic. If necessary, we can add a constant to the test function, and since the PDE involves only derivatives of the test function, this does not change the viscosity inequality.

10. Geometric interpretation

The definition is best understood geometrically.

- If a smooth function touches u **from above**, then u must satisfy the **subsolution inequality** there.
- If a smooth function touches u **from below**, then u must satisfy the **supersolution inequality** there.

Thus viscosity theory translates the PDE into a condition on **one-sided local tangency**.

This is the correct framework for non-smooth value functions in optimal control.

For the eikonal equation, the orientation of a cusp matters. At the downward cusp of $1 - |x|$, an upper test function may have slope $p = 0$, which satisfies the subsolution inequality $|p| \leq 1$; no lower test is available at the cusp, so the supersolution condition is vacuous there. At the upward cusp of $|x|$, however, a lower test with slope $p = 0$ exists and violates the required supersolution inequality $|p| \geq 1$.

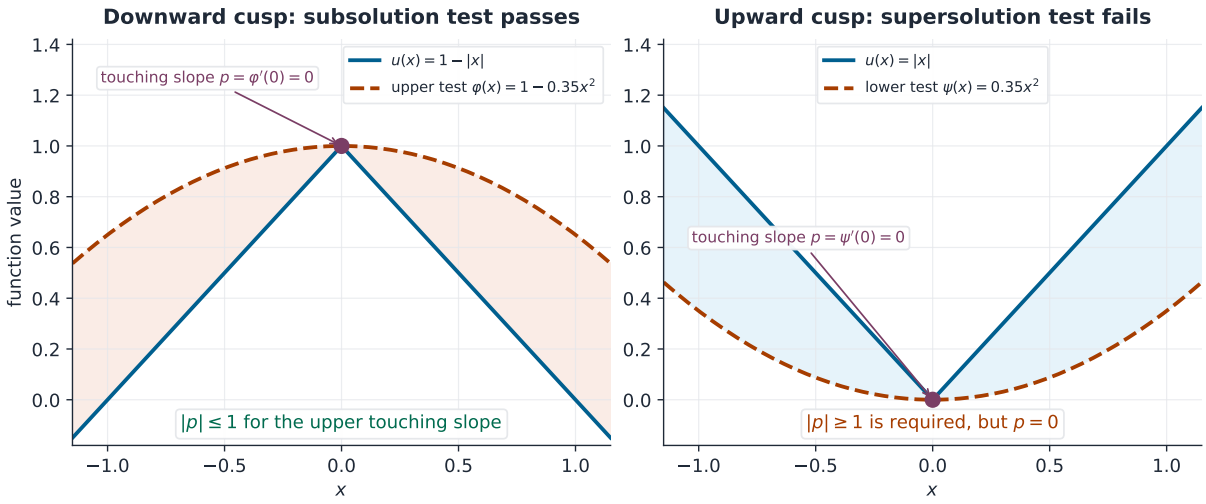


Figure 6.1. Viscosity inequalities distinguish cusp orientation. The downward cusp passes the one-sided tests for $|u'| - 1 = 0$, whereas the upward cusp fails the supersolution test at the origin.

11. Example 6.1: verification for the eikonal equation

Consider

$$|u'(x)| = 1 \quad \text{in } (-1, 1), \qquad u(-1) = u(1) = 0.$$

We claim that

$$u(x) = 1 - |x|$$

is a viscosity solution.

We verify this directly using the test-function definition. For the stationary equation $|u'| - 1 = 0$, there is no time derivative:

- if $\varphi \in C^1$ touches u from above at an interior point x_0, then we must show $|\varphi'(x_0)| \le 1$;
- if $\psi \in C^1$ touches u from below at x_0, then we must show $|\psi'(x_0)| \ge 1$.

We normalize each test function to agree with u at the touching point. By §9.2, ordinary local maxima and minima can be used in these tests.

Step 1. Classical points

If $x \neq 0$, then u is differentiable and

$$u'(x) = \begin{cases} 1, & x < 0, \\ -1, & x > 0. \end{cases}$$

Hence

$$|u'(x)| = 1.$$

So the equation holds classically away from $x = 0$.

If a C^1 test function touches u from either side at such a point, the derivative of their difference is zero. Hence the derivative of the test function equals $u'(x)$, and both viscosity inequalities hold.

Step 2. Subsolution test at $x = 0$

Let $\varphi \in C^1$ touch u from above at 0, with $\varphi(0) = u(0) = 1$. Then, for all sufficiently small h,

$$1 - |h| = u(h) \le \varphi(h).$$

Subtracting $\varphi(0) = 1$ gives

$$\varphi(h) - \varphi(0) \ge -|h|.$$

For $h > 0$, divide by h and let $h \downarrow 0$:

$$\frac{\varphi(h) - \varphi(0)}{h} \ge -1 \quad \Longrightarrow \quad \varphi'(0) \ge -1.$$

For $h < 0$, division by h reverses the inequality. Letting $h \uparrow 0$, we obtain

$$\frac{\varphi(h) - \varphi(0)}{h} \le 1 \quad \Longrightarrow \quad \varphi'(0) \le 1.$$

Therefore every upper test function satisfies

$$-1 \le \varphi'(0) \le 1, \qquad \text{so} \qquad |\varphi'(0)| - 1 \le 0.$$

This proves the viscosity subsolution inequality at the cusp.

Step 3. Supersolution test at $x = 0$

Suppose that a C^1 function ψ touches u from below at 0, with $\psi(0) = 1$. Then

$$\psi(h) - \psi(0) \le -|h|$$

for all sufficiently small h.

For $h > 0$, division by h and passage to the limit give

$$\psi'(0) \le -1.$$

For $h < 0$, the inequality reverses after division by h, giving

$$\psi'(0) \ge 1.$$

These two conditions are incompatible. Thus no C^1 test function can touch u from below at the cusp.

The supersolution condition asks for an inequality whenever a lower test function exists. Since no such test function exists at 0, the condition is satisfied there automatically.

Step 4. Boundary values and conclusion

Both viscosity inequalities hold at every point of $(-1, 1)$. Moreover, u is continuous on $[-1, 1]$ and

$$u(-1) = u(1) = 0.$$

Therefore $u(x) = 1 - |x|$ is a viscosity solution of the eikonal equation with the stated boundary values.

12. Example 6.2: a sawtooth profile is not a viscosity solution

In Week 5, we saw that the same equation may admit many almost everywhere solutions, including sawtooth-type functions. A piecewise linear sawtooth profile with slopes ± 1 satisfies the equation away from corner points. We now show that an interior **lower corner** prevents it from being a viscosity solution.

Let $x_0 \in (-1, 1)$ be such a lower corner. Near x_0, the profile has the form

$$u(x) = u(x_0) + |x - x_0|.$$

Choose the constant test function

$$\psi(x) = u(x_0).$$

Then $\psi \in C^1$, $\psi(x_0) = u(x_0)$, and

$$u(x) - \psi(x) = |x - x_0|$$

has a strict local minimum at x_0. Thus ψ is an admissible lower test function.

The viscosity supersolution condition would require

$$|\psi'(x_0)| - 1 \geq 0.$$

But $\psi'(x_0) = 0$, so

$$|\psi'(x_0)| - 1 = -1 < 0.$$

Therefore the sawtooth profile is **not** a viscosity solution.

One lower test function violating the supersolution inequality is enough to reach this conclusion.

This shows that the viscosity notion correctly selects the admissible solution and rules out spurious almost everywhere solutions.

13. Why this matters for optimal control

We now return to the logic of the course.

From optimal control, we obtained:

- the controlled dynamics,
- the cost functional,
- the value function,
- and the dynamic programming principle.

From these, we derived Hamilton–Jacobi equations.

However, the value function is generally non-smooth, so the classical PDE interpretation is not sufficient. The viscosity solution framework resolves exactly this difficulty.

Therefore, viscosity solutions are the correct PDE language for value functions in optimal control.

14. Summary

The vanishing viscosity method regularizes the Hamilton–Jacobi equation by adding a small Laplacian:

$$u_t^\varepsilon + H(x, Du^\varepsilon) = \varepsilon \Delta u^\varepsilon.$$

Because the regularized equation is parabolic, the maximum principle applies.
Passing formally to the limit suggests the viscosity inequalities:

- touching from above gives the subsolution condition;
- touching from below gives the supersolution condition.

This leads to the definition of viscosity subsolutions, supersolutions, and viscosity solutions.

The central benefit of this theory is that it allows us to work with non-smooth functions while preserving the correct PDE structure. In particular:

- the function $u(x) = 1 - |x|$ is a viscosity solution of the one-dimensional eikonal equation;
- sawtooth-type almost everywhere solutions are excluded.

Thus viscosity solutions provide the correct selection principle for Hamilton–Jacobi equations, especially those arising from optimal control.

15. Exercises

Exercise 6.1

Determine whether

$$u(x) = |x|$$

is a viscosity solution of

$$|u'| = 1 \quad \text{in } (-1, 1).$$

Exercise 6.2

Show that if $u - \varphi$ has a local maximum at (x_0, t_0), then by adding a constant to φ, one may assume

$$u(x_0, t_0) = \varphi(x_0, t_0).$$

Exercise 6.3

Explain why the definition is unchanged if we replace “strict local maximum/minimum” by “local maximum/minimum.”

Exercise 6.4

Verify that if $u \in C^1$, then the viscosity definition reduces to the classical definition.

Exercise 6.5

Write a short explanation of why viscosity solutions are the natural notion of solution for value functions in optimal control.