Week 7. Generalized Differentials

Primary reference: Tran, *Hamilton–Jacobi Equations: Theory and Applications*, §1.2.3 (Definition 1.7, Theorem 1.8, Definition 1.9, Proposition 1.10)

1. Why We Study Generalized Differentials at This Point

In the textbook, generalized differentials are introduced early as part of the PDE framework. In this course, however, we first developed the optimal control viewpoint, then derived the Hamilton–Jacobi–Bellman equation from the Dynamic Programming Principle, and only after seeing the failure of classical differentiability did we introduce viscosity solutions.

This lecture now returns to the first-order analytic structure behind viscosity solutions. The main point is simple: value functions arising from optimal control are often Lipschitz continuous but not differentiable everywhere. Therefore, if we want to interpret a first-order equation such as

$$F(x, u, Du) = 0$$

at nonsmooth points, we need a substitute for the classical gradient. That substitute is provided by the **superdifferential** and **subdifferential**.

These generalized differentials encode the admissible first-order slopes of a continuous function at a point, and they give an equivalent and very efficient formulation of viscosity solutions.

2. Generalized Differentials

Let $\Omega \subset \mathbb{R}^n$ be open and let $u \in C(\Omega)$. For $x \in \Omega$, we define:

Superdifferential

$$D^+u(x) = \left\{ p \in \mathbb{R}^n : \limsup_{y \to x} \frac{u(y) - u(x) - p \cdot (y - x)}{|y - x|} \leq 0 \right\}.$$

Subdifferential

$$D^-u(x) = \left\{ p \in \mathbb{R}^n : \liminf_{y \to x} \frac{u(y) - u(x) - p \cdot (y - x)}{|y - x|} \geq 0 \right\}.$$

These are called the **generalized differentials** of u at x.

3. Geometric Meaning

The set $D^+u(x)$ consists of all first-order slopes that support u **from above** at x, up to an $o(|y-x|)$ error. In other words, if $p \in D^+u(x)$, then near x,

$$u(y) \leq u(x) + p \cdot (y - x) + o(|y - x|).$$

Similarly, the set $D^-u(x)$ consists of all first-order slopes that support u **from below** at x, meaning that near x,

$$u(y) \geq u(x) + p \cdot (y - x) + o(|y - x|).$$

Thus:

- $D^+u(x)$ corresponds to **upper supporting affine functions**,
- $D^-u(x)$ corresponds to **lower supporting affine functions**.

If u is differentiable at x, then both sets reduce to the single vector $Du(x)$. Hence generalized differentials extend the ordinary derivative to the nonsmooth setting.

4. Theorem 1.8: Equivalence with Touching Test Functions

A central result is the following.

Theorem 1.8

Let $f \in C(\Omega)$, $x \in \Omega$, and $p \in \mathbb{R}^n$. Then

$$p \in D^+ f(x) \quad \Longleftrightarrow \quad \exists \varphi \in C^1(\Omega) \text{ such that } D\varphi(x) = p \text{ and } f - \varphi \text{ has a local maximum at } x.$$

Similarly,

$$p \in D^- f(x) \quad \Longleftrightarrow \quad \exists \varphi \in C^1(\Omega) \text{ such that } D\varphi(x) = p \text{ and } f - \varphi \text{ has a local minimum at } x.$$

This theorem is the precise bridge between:

- the **test function** formulation of viscosity solutions, and
- the **generalized differential** formulation.

So the two viewpoints are not merely related; they are mathematically equivalent.

5. Proof Idea of Theorem 1.8

We explain the key step in the implication

$$p \in D^+ f(x) \Longrightarrow \exists \varphi \in C^1 \text{ touching from above at } x.$$

Assume $p \in D^+ f(x)$. Define

$$\sigma(r) = \max\left\{0, \sup_{\substack{y \in B_r(x) \\ y \neq x}} \frac{f(y) - f(x) - p \cdot (y - x)}{|y - x|}\right\}.$$

Then $\sigma(r) \geq 0$ and $\sigma(r) \to 0$ as $r \downarrow 0$.

Next define

$$\rho(r) = \int_r^{2r} \sigma(s)\, ds.$$

This function plays the role of a smooth radial perturbation. It satisfies:

1. $\rho(r) \geq r\,\sigma(r)$,
2. $\rho(r) = o(r)$ as $r \downarrow 0$,
3. the function $y \mapsto \rho(|y - x|)$ is regular enough to produce a C^1 touching function at x.

Now define

$$\varphi(y) = f(x) + p \cdot (y - x) + \rho(|y - x|).$$

By construction, near x,

$$f(y) \leq \varphi(y), \qquad f(x) = \varphi(x),$$

so $f - \varphi$ has a local maximum at x. Moreover, since $\rho(r) = o(r)$,

$$D\varphi(x) = p.$$

This proves the existence of a C^1 test function touching from above.

The converse implication is easier: if $\varphi \in C^1$ touches f from above at x, then Taylor expansion of φ implies immediately that $D\varphi(x) \in D^+ f(x)$.

6. Equivalent Definition of Viscosity Solutions

Consider the first-order stationary equation

$$F(x, u(x), Du(x)) = 0 \qquad \text{in } \Omega,$$

where F is continuous.

Viscosity subsolution

A function $u \in C(\Omega)$ is a viscosity subsolution if

$$F(x, u(x), p) \leq 0 \qquad \text{for every } x \in \Omega \text{ and every } p \in D^+u(x).$$

Viscosity supersolution

A function $u \in C(\Omega)$ is a viscosity supersolution if

$$F(x, u(x), p) \geq 0 \qquad \text{for every } x \in \Omega \text{ and every } p \in D^-u(x).$$

Viscosity solution

A function is a viscosity solution if it is both a viscosity subsolution and a viscosity supersolution.

This is exactly equivalent to the test-function definition introduced earlier. The advantage of the present formulation is that once $D^+u(x)$ and $D^-u(x)$ are known, checking the viscosity inequalities becomes much more direct.

7. Basic Properties of Generalized Differentials

We now list the main structural facts.

Proposition 1.10

Let $f : \Omega \to \mathbb{R}$ be continuous and let $x \in \Omega$. Then:

(a) Sign symmetry

$$D^+f(x) = -D^-(-f)(x).$$

(b) Convexity

Both $D^+f(x)$ and $D^-f(x)$ are convex sets (possibly empty).

(c) Characterization of differentiability

The sets $D^+f(x)$ and $D^-f(x)$ are both nonempty if and only if f is differentiable at x. In that case,

$$D^+f(x) = D^-f(x) = \{Df(x)\}.$$

(d) Density

If we define

$$\Omega^+ := \{x \in \Omega : D^+f(x) \neq \emptyset\}, \qquad \Omega^- := \{x \in \Omega : D^-f(x) \neq \emptyset\},$$

then both Ω^+ and Ω^- are dense in Ω.

These facts show that generalized differentials are not ad hoc objects. They behave in a systematic way and recover the classical derivative whenever upper and lower first-order supports coexist.

8. Example 7.1: Basic Computations

We now compute generalized differentials in simple one-dimensional examples.

Example 7.1(a): $u(x) = |x|$ at $x = 0$

At $x = 0$, the graph of u has a sharp **minimum**. Therefore lower supports exist, but upper supports do not.

We have

$$D^+u(0) = \emptyset, \qquad D^-u(0) = [-1, 1].$$

Indeed, every slope $p \in [-1, 1]$ gives a line that lies below the graph of $|x|$ to first order, while no C^1 function can touch $|x|$ from above at the cusp with the same first-order behavior.

Example 7.1(b): $u(x) = -|x|$ at $x = 0$

Now the graph has a sharp **maximum**. Therefore upper supports exist, but lower supports do not.

We have

$$D^+u(0) = [-1, 1], \qquad D^-u(0) = \emptyset.$$

This is the exact sign-reversed version of the previous example.

The families of supporting lines make the two generalized differentials visible. At an upward cusp, every slope in $[-1, 1]$ supports the graph from below but none supports it from above. At a downward cusp, the roles are reversed.

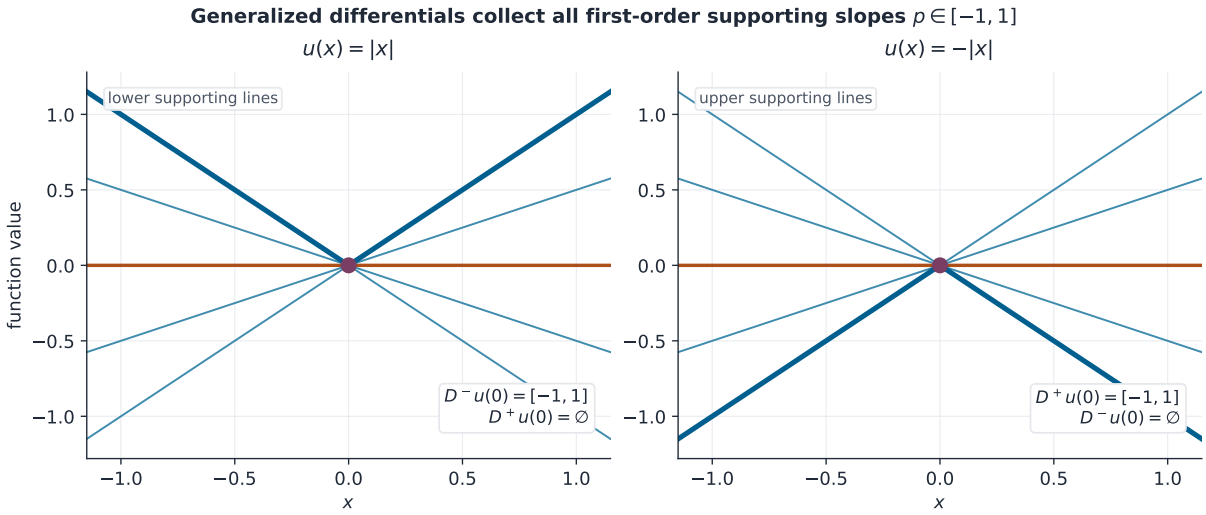


Figure 7.1. Left: the lower supporting slopes of $|x|$ fill $D^-u(0) = [-1, 1]$, while $D^+u(0)$ is empty. Right: for $-|x|$, the same interval is $D^+u(0)$, while $D^-u(0)$ is empty.

Example 7.1(c): $u(x) = \mathbf{1}_{(0,\infty)}(x)$

This function is discontinuous at $x = 0$. Since the viscosity framework discussed here is formulated for continuous functions, this example lies outside the present setting.

The formulas defining generalized differentials can still be evaluated: with $u(0) = 0$,

$$D^+u(0) = \emptyset, \qquad D^-u(0) = [0, \infty).$$

However, Tran's Definition 1.9 does not apply because u is not continuous. The point of the example is to remember that continuity is part of the basic framework used here.

9. Example 7.2: The Eikonal Equation and Cusp Selection

Consider

$$|Du| = 1 \qquad \text{in } B_1(0) \subset \mathbb{R}^n.$$

A natural question is whether the nondifferentiable function

$$u(x) = |x|$$

is a viscosity solution.

The answer is **no**.

At $x = 0$,

$$D^+u(0) = \emptyset, \qquad D^-u(0) = \overline{B(0,1)}.$$

To be a viscosity supersolution of $|Du| = 1$, we would need

$$|p| - 1 \geq 0 \qquad \text{for every } p \in D^-u(0).$$

But this is false, since $0 \in \overline{B(0,1)}$ and

$$|0| - 1 = -1 < 0.$$

Hence $u(x) = |x|$ is **not** a viscosity solution of $|Du| = 1$.

This example is important because it shows that viscosity solutions distinguish between two cusp profiles that both satisfy the equation almost everywhere.

By contrast, the function

$$u(x) = 1 - |x|$$

is the correct viscosity solution of

$$|Du| = 1 \quad \text{in } (-1,1), \qquad u(\pm 1) = 0,$$

and more generally the distance-to-the-boundary profile is the correct geometric solution for the Dirichlet eikonal problem.

10. Example 7.3: The Distance Function

Let $U \subset \mathbb{R}^n$ be a bounded smooth domain and define

$$u(x) = \operatorname{dist}(x, \partial U), \qquad x \in U.$$

We claim that u is a viscosity solution of

$$|Du| = 1 \qquad \text{in } U, \qquad u = 0 \qquad \text{on } \partial U.$$

Step 1. Subsolution property

The distance function is 1-Lipschitz:

$$|u(x) - u(y)| \leq |x - y|.$$

A standard consequence is that for every $p \in D^+u(x)$,

$$|p| \leq 1.$$

Therefore

$$|p| - 1 \leq 0 \qquad \text{for all } p \in D^+u(x),$$

so u is a viscosity subsolution.

Step 2. Supersolution property

Take $x \in U$ and $p \in D^- u(x)$. Let $z \in \partial U$ be a nearest boundary point, so

$$u(x) = |x - z|.$$

Set

$$e = \frac{x - z}{|x - z|}.$$

For $0 < h < u(x) = \operatorname{dist}(x, \partial U)$, let

$$y = x - he.$$

Since $B(x, u(x)) \subset U$, we have $y \in U$, even when U is not convex, and

$$u(y) \le |y - z| = |x - z| - h = u(x) - h.$$

On the other hand, since $p \in D^- u(x)$,

$$u(y) \ge u(x) + p \cdot (y - x) + o(|y - x|) = u(x) - h\, p \cdot e + o(h).$$

Combining the two inequalities gives

$$u(x) - h \ge u(x) - h\, p \cdot e + o(h).$$

Divide by h and let $h \downarrow 0$. Then

$$p \cdot e \ge 1.$$

Hence

$$|p| \ge 1.$$

Therefore

$$|p| - 1 \ge 0 \qquad \text{for all } p \in D^- u(x),$$

so u is a viscosity supersolution.

Thus $u(x) = \operatorname{dist}(x, \partial U)$ is a viscosity solution of the eikonal equation with zero boundary data.

Later, after the comparison principle is established, this solution will also be seen to be unique.

11. Connection with Optimal Control

Although generalized differentials are introduced in the PDE language, their role is especially natural from the viewpoint of optimal control.

The value function of a control problem may fail to be differentiable because:

- there may be several competing optimal trajectories,
- switching may occur between different optimal controls,
- geometric constraints may produce corners or kinks.

At such points, there is no single classical gradient. Nevertheless, there are still meaningful first-order supporting slopes, and these are exactly the elements of $D^+ u(x)$ and $D^- u(x)$.

Therefore generalized differentials provide the correct first-order language for nonsmooth value functions, and the viscosity inequalities express the HJB equation in terms of these admissible generalized slopes.

11.1 From the DPP to a Viscosity Solution

We now prove the result stated in Week 4, §9 (Tran, §2.3, Theorem 2.11): the discounted optimal-control value function is a viscosity solution of the stationary HJB equation. The Dynamic Programming Principle was proved in Week 3, and the viscosity definition was introduced in Week 6. We can therefore complete the argument without assuming that the value function is differentiable.

We use the standing assumptions from Weeks 2–4:

- V is a nonempty compact metric space;
- $b : \mathbb{R}^n \times V \to \mathbb{R}^n$ and $f : \mathbb{R}^n \times V \to \mathbb{R}$ are continuous;
- $|b(x,v)| \le M_b$ and $|f(x,v)| \le M_f$ for all (x,v);
- b and f are Lipschitz continuous in x, uniformly in v, with constants L_b and L_f;
- $\lambda > 0$.

For a measurable control $v(\cdot) : [0,\infty) \to V$, let

$$y'(s) = b(y(s), v(s)), \qquad y(0) = x.$$

The cost and the value function are

$$J(x, v(\cdot)) = \int_0^\infty e^{-\lambda s} f(y(s), v(s))\, ds, \qquad u(x) = \inf_{v(\cdot)} J(x, v(\cdot)).$$

We retain Tran's sign convention:

$$H(x,p) = \sup_{v \in V} \{-b(x,v) \cdot p - f(x,v)\}.$$

First, u is bounded and uniformly continuous. Boundedness follows from

$$|u(x)| \le \frac{M_f}{\lambda}.$$

For continuity, compare trajectories from x and z using the same control. The trajectory estimate from Week 2 gives

$$|y_x(s) - y_z(s)| \le e^{L_b s}|x - z|.$$

Splitting the cost at any $R > 0$, and then taking infima, gives

$$|u(x) - u(z)| \le L_f |x - z| \int_0^R e^{(L_b - \lambda)s}\, ds + \frac{2M_f}{\lambda} e^{-\lambda R}.$$

Choose R large and then $|x-z|$ small. This proves uniform continuity for every $\lambda > 0$; the stronger condition $\lambda > L_b$ is not needed here.

11.2 The Stationary Viscosity Statement

Theorem (Tran, Theorem 2.11). Under the assumptions above, the value function u is a viscosity solution of

$$\boxed{\lambda u + H(x, Du) = 0 \qquad \text{in } \mathbb{R}^n.}$$

This is the viscosity definition for $F(x,r,p) = \lambda r + H(x,p)$. For this stationary equation, the test-function definition requires the following two inequalities. Let φ be C^1 near x_0, and normalize it by adding a constant so that $\varphi(x_0) = u(x_0)$.

- If $u - \varphi$ has a local maximum at x_0, then
$$\lambda u(x_0) + H(x_0, D\varphi(x_0)) \le 0.$$
- If $u - \varphi$ has a local minimum at x_0, then
$$\lambda u(x_0) + H(x_0, D\varphi(x_0)) \ge 0.$$

As discussed in Week 6, strictness of the local extremum is not required. The proof below treats these two cases separately.

11.3 Short-Time DPP and the Test-Function Identity

For every $h > 0$, the DPP gives

$$u(x_0) = \inf_{v(\cdot)} \left\{ \int_0^h e^{-\lambda s} f(y(s), v(s))\, ds + e^{-\lambda h} u(y(h)) \right\}, \qquad y(0) = x_0.$$

Here the infimum can be taken over measurable controls on $[0, h]$, since each such control can be extended beyond h.

The velocity bound gives, for every admissible control,

$$|y(s) - x_0| \le M_b s.$$

Consequently, if h is sufficiently small, every trajectory remains in the neighborhood where the test function touches u. This estimate is uniform in the control.

Although u need not be differentiable, the chain rule applies to φ along each absolutely continuous trajectory:

$$e^{-\lambda h}\varphi(y(h)) - \varphi(x_0) = \int_0^h e^{-\lambda s} \left[b(y(s), v(s)) \cdot D\varphi(y(s)) - \lambda \varphi(y(s)) \right] ds.$$

For this test function, write

$$q_\varphi(x, v) = f(x, v) + b(x, v) \cdot D\varphi(x) - \lambda \varphi(x).$$

Since $\varphi(x_0) = u(x_0)$, the identity becomes

$$\begin{aligned} &\int_0^h e^{-\lambda s} f(y(s), v(s))\, ds + e^{-\lambda h}\varphi(y(h)) - u(x_0) \\ &\qquad = \int_0^h e^{-\lambda s} q_\varphi(y(s), v(s))\, ds. \end{aligned}$$

This identity replaces the formal Taylor expansion of the value function used in Week 4.

11.4 Proof of the Subsolution Property

Suppose that $u - \varphi$ has a local maximum at x_0, with $u(x_0) = \varphi(x_0)$. Then

$$u(x) \le \varphi(x) \qquad \text{for } x \text{ near } x_0.$$

Fix any $v \in V$, use the constant control $v(s) \equiv v$, and denote its trajectory by y_v. The DPP and the upper touching property give, for small $h > 0$,

$$\begin{aligned} u(x_0) &\le \int_0^h e^{-\lambda s} f(y_v(s), v)\, ds + e^{-\lambda h} u(y_v(h)) \\ &\le \int_0^h e^{-\lambda s} f(y_v(s), v)\, ds + e^{-\lambda h}\varphi(y_v(h)). \end{aligned}$$

The test-function identity therefore implies

$$0 \le \int_0^h e^{-\lambda s} q_\varphi(y_v(s), v)\, ds.$$

Divide by h and let $h \downarrow 0$. By continuity,

$$0 \le f(x_0, v) + b(x_0, v) \cdot D\varphi(x_0) - \lambda u(x_0).$$

This holds for every $v \in V$. Taking the supremum after rearranging yields

$$\lambda u(x_0) + \sup_{v \in V}\{-b(x_0, v) \cdot D\varphi(x_0) - f(x_0, v)\} \leq 0.$$

Thus

$$\boxed{\lambda u(x_0) + H(x_0, D\varphi(x_0)) \leq 0.}$$

11.5 Proof of the Supersolution Property

Suppose that $u - \varphi$ has a local minimum at x_0, with $u(x_0) = \varphi(x_0)$. Then

$$u(x) \geq \varphi(x) \qquad \text{for } x \text{ near } x_0.$$

The infimum in the DPP need not be attained. For each sufficiently small $h > 0$, choose an h^2-optimal control $v_h(\cdot)$ for the DPP expression, with trajectory y_h, so that

$$\int_0^h e^{-\lambda s} f(y_h(s), v_h(s))\, ds + e^{-\lambda h} u(y_h(h)) \leq u(x_0) + h^2.$$

Using the lower touching property and the test-function identity, we obtain

$$\begin{aligned}\int_0^h e^{-\lambda s} q_\varphi(y_h(s), v_h(s))\, ds &= \int_0^h e^{-\lambda s} f(y_h(s), v_h(s))\, ds + e^{-\lambda h}\varphi(y_h(h)) - u(x_0) \\ &\leq h^2.\end{aligned}$$

The control may depend on h, so we need an estimate that is uniform in v. On a closed ball around x_0 contained in the touching neighborhood, continuity of q_φ and compactness of V give

$$\omega(h) := \sup_{\substack{|x - x_0| \leq M_b h \\ v \in V}} |q_\varphi(x, v) - q_\varphi(x_0, v)| \longrightarrow 0 \qquad \text{as } h \downarrow 0.$$

Set

$$m = \inf_{v \in V} q_\varphi(x_0, v) = -\lambda u(x_0) - H(x_0, D\varphi(x_0)).$$

Since $|y_h(s) - x_0| \leq M_b h$ for $0 \leq s \leq h$, we have

$$q_\varphi(y_h(s), v_h(s)) \geq m - \omega(h).$$

It follows that

$$(m - \omega(h)) \int_0^h e^{-\lambda s}\, ds \leq \int_0^h e^{-\lambda s} q_\varphi(y_h(s), v_h(s))\, ds \leq h^2.$$

Therefore

$$m \leq \omega(h) + \frac{\lambda h^2}{1 - e^{-\lambda h}}.$$

Letting $h \downarrow 0$ gives $m \leq 0$, or equivalently,

$$\boxed{\lambda u(x_0) + H(x_0, D\varphi(x_0)) \geq 0.}$$

Together with the subsolution property, this proves the theorem.

The error h^2 was chosen because it is $o(h)$: it vanishes after division by the length of the time interval. No optimal control or convergence of the controls v_h is required.

11.6 Formulation with Generalized Differentials

By Theorem 1.8, the result just proved is equivalent to

$$\lambda u(x) + H(x,p) \le 0 \qquad \text{for every } p \in D^+u(x),$$

and

$$\lambda u(x) + H(x,p) \ge 0 \qquad \text{for every } p \in D^-u(x).$$

This gives the promised connection between optimal control and the generalized differentials studied in this lecture. The DPP supplies the inequalities, and smooth touching functions allow us to express them without differentiating u.

At a differentiability point, $D^+u(x) = D^-u(x) = \{Du(x)\}$, so the two inequalities give the classical equality

$$\lambda u(x) + H(x, Du(x)) = 0.$$

The argument proves that the control-defined value function is a viscosity solution. Uniqueness will follow later from the comparison principle.

11.7 Relation to Vanishing Viscosity

There are two distinct arguments in the course:

- **DPP argument:** start with the value function defined by the control problem and prove directly that it satisfies the viscosity inequalities. This is the argument completed above.
- **Vanishing viscosity argument:** start with smooth solutions of a regularized PDE and show that a locally uniform limit satisfies the viscosity inequalities for the first-order equation. Week 6 introduced the mechanism; Week 8, §4, gives the existence argument under uniform estimates.

For example, in the time-dependent setting of Week 8,

$$u^\varepsilon_t + H(x, Du^\varepsilon) = \varepsilon \Delta u^\varepsilon.$$

Each u^ε solves this regularized equation. The vanishing viscosity conclusion concerns the limit: if $u^{\varepsilon_j} \to w$ locally uniformly, then

$$w_t + H(x, Dw) = 0$$

in the viscosity sense. This limiting argument by itself does not identify w with a particular control value function. Such an identification requires the same PDE and data, a viscosity characterization of the value function, and a comparison principle.

12. Summary

In this lecture we introduced the generalized differential sets $D^+u(x)$ and $D^-u(x)$, which extend the notion of gradient to continuous but possibly nondifferentiable functions.

Theorem 1.8 showed that membership in $D^+u(x)$ or $D^-u(x)$ is equivalent to the existence of a C^1 test function touching from above or below. This gives a direct link to the test-function definition of viscosity solutions.

Definition 1.9 then reformulated viscosity subsolutions and supersolutions entirely in terms of generalized differentials.

Finally, Proposition 1.10 described the basic structure of these sets, and the examples showed how they are used in practice, especially for cusp profiles and the eikonal equation.

The key lesson is that generalized differentials are the correct first-order objects for Hamilton–Jacobi equations when classical differentiability fails.

13. Exercises

Exercise 7.1

Compute $D^+u(x)$ and $D^-u(x)$ for every $x \in \mathbb{R}$ when

$$u(x) = |x| \qquad \text{and} \qquad u(x) = -|x|.$$

Exercise 7.2

Prove directly from the definitions that

$$D^+u(x) = -D^-(-u)(x).$$

Then show that both $D^+u(x)$ and $D^-u(x)$ are convex.

Exercise 7.3

Let $u \in C(\Omega)$ and assume that both $D^+u(x)$ and $D^-u(x)$ are nonempty. Prove that u is differentiable at x and that

$$D^+u(x) = D^-u(x) = \{Du(x)\}.$$

Exercise 7.4

For $u(x) = |x|$ in $B_1(0) \subset \mathbb{R}^n$, compute the generalized differentials at the origin and prove that u is not a viscosity solution of

$$|Du| = 1 \qquad \text{in } B_1(0).$$

Exercise 7.5

Let $U \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Prove that

$$u(x) = \operatorname{dist}(x, \partial U)$$

is 1-Lipschitz and verify both viscosity inequalities for

$$|Du| = 1 \qquad \text{in } U, \qquad u = 0 \qquad \text{on } \partial U.$$

Exercise 7.6

In the construction used in Theorem 1.8, verify that

$$\rho(r) = \int_r^{2r} \sigma(s)\, ds$$

satisfies

$$r\sigma(r) \le \rho(r) \le r\sigma(2r), \qquad \frac{\rho(r)}{r} \to 0 \quad \text{as } r \downarrow 0.$$

Explain why these estimates give $D\varphi(x) = p$ for the touching function constructed in the proof.

14. References

1. H. V. Tran, *Hamilton–Jacobi Equations: Theory and Applications*, AMS GSM 213, 2021.