Week 9. Static Comparison Principle and Perron's Method

Tran: §1.5 (Theorem 1.18), §1.8 (Theorems 1.30 and 1.31)

1. Overview

Following Tran §1.5, we consider the static equation with coefficient 1 in front of u:

$$u + H(x, Du) = 0 \qquad \text{in } \mathbb{R}^n.$$

Why this week matters

In the first part of the course, we introduced optimal control, the Dynamic Programming Principle, and the derivation of the Hamilton–Jacobi–Bellman equation. Week 7 proved directly from the DPP that the discounted value function is a viscosity solution, and Week 8 established existence for the Cauchy problem by vanishing viscosity under suitable assumptions. We now prove **comparison and uniqueness** for the static problem and study **Perron's method** as another existence construction.

- The **comparison principle** yields uniqueness.
- **Perron's method** yields existence by constructing the largest admissible subsolution.

Together, they form the basic well-posedness theory for first-order static Hamilton–Jacobi equations.

2. Learning Objectives

By the end of this lecture, students should be able to:

1. state the static comparison principle and explain why it implies uniqueness;
2. carry out the main estimates in the doubling-of-variables argument;
3. explain the roles of the diagonal and coercive penalization terms;
4. distinguish Perron's analytic existence method from vanishing viscosity;
5. state the correct stability properties of subsolutions and supersolutions;
6. verify the eikonal distance solution in the viscosity sense.

Notation and function spaces

As in Weeks 6–8, $H(x, p)$ denotes the Hamiltonian. The full PDE expression for the static equation is

$$F(x, r, p) = r + H(x, p).$$

We use φ for an **upper test function** and ψ for a **lower test function**. At a normalized contact point x_0,

- $u - \varphi$ has a local maximum and $u(x_0) = \varphi(x_0)$: the subsolution inequality is $u(x_0) + H(x_0, D\varphi(x_0)) \le 0$;
- $u - \psi$ has a local minimum and $u(x_0) = \psi(x_0)$: the supersolution inequality is $u(x_0) + H(x_0, D\psi(x_0)) \ge 0$.

Ordinary local extrema are sufficient by Week 6, §9.2. Adding a constant to a test function enforces equality at the contact point without changing its derivatives, as in Week 8, §3. The value argument of F remains the value of the function being tested.

Recall that BUC means bounded uniformly continuous, and Lip means globally Lipschitz. We write $\mathrm{Lip}(v) \le L$ when $|v(x) - v(y)| \le L|x - y|$ for every x, y.

3. Lipschitz Assumptions on H (Tran (1.13), p.16)

Assume that there exists $C > 0$ such that for all $x, y, p, q \in \mathbb{R}^n$,

$$|H(x, p) - H(x, q)| \le C|p - q|, \qquad |H(x, p) - H(y, p)| \le C(1 + |p|)|x - y|.$$

The first condition is a global Lipschitz assumption in the momentum variable p, and the second is a spatial Lipschitz assumption with at most linear growth in $|p|$. These assumptions are sufficient for the standard comparison proof.

Remark.
The first assumption is convenient but restrictive. For example,

$$H(x,p) = \frac{1}{2}|p|^2 + V(x)$$

is not globally Lipschitz in p. In §7, we will see how a Lipschitz bound on the functions being compared allows us to use uniform continuity of H on bounded p-regions.

4. Comparison Principle (Tran Theorem 1.18, p.16)

Assume (1.13). Let $u, v \in \mathrm{BUC}(\mathbb{R}^n)$. If u is a viscosity subsolution and v is a viscosity supersolution of the static equation, then

$$u(x) \le v(x) \qquad \text{for all } x \in \mathbb{R}^n.$$

This is the basic uniqueness mechanism in viscosity theory: every subsolution must lie below every supersolution.

5. Proof: The Doubling-of-Variables Argument (Tran pp.17–19)

Assume by contradiction that

$$\sup_{x\in\mathbb{R}^n} \big(u(x) - v(x)\big) = \sigma > 0.$$

Choose $x_1 \in \mathbb{R}^n$ such that

$$u(x_1) - v(x_1) > \frac{3\sigma}{4}.$$

For $\varepsilon > 0$ sufficiently small, define

$$\Phi_\varepsilon(x,y) = u(x) - v(y) - \frac{|x-y|^2}{\varepsilon^2} - \varepsilon(|x|^2 + |y|^2).$$

Since u and v are bounded and uniformly continuous, and since the last two penalty terms force $\Phi_\varepsilon(x,y) \to -\infty$ as $|x| + |y| \to \infty$, the function Φ_ε attains a global maximum at some point

$$(x_\varepsilon, y_\varepsilon) \in \mathbb{R}^n \times \mathbb{R}^n.$$

The structure of Φ_ε is fundamental:

- the term $-|x-y|^2/\varepsilon^2$ forces x and y to be close,
- the term $-\varepsilon(|x|^2 + |y|^2)$ ensures that a maximizing pair exists for each fixed $\varepsilon > 0$.

We now split the proof into five steps.

Step 1. Estimates on the maximizing pair

By maximality,

$$\Phi_\varepsilon(x_\varepsilon, y_\varepsilon) \ge \Phi_\varepsilon(x_1, x_1) = u(x_1) - v(x_1) - 2\varepsilon|x_1|^2.$$

Choosing ε small enough, we obtain

$$\Phi_\varepsilon(x_\varepsilon, y_\varepsilon) \ge \frac{\sigma}{2}.$$

Comparing also with $\Phi_\varepsilon(0,0)$ and using boundedness of u, v, one finds

$$\frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon^2} + \varepsilon(|x_\varepsilon|^2 + |y_\varepsilon|^2) \le C,$$

hence

$$|x_\varepsilon - y_\varepsilon| \le C\varepsilon, \qquad |x_\varepsilon| + |y_\varepsilon| \le \frac{C}{\sqrt{\varepsilon}}.$$

Step 2. Vanishing of the main penalty term

We claim that

$$\frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon^2} \to 0 \qquad \text{as } \varepsilon \to 0.$$

Indeed, by maximality,

$$\Phi_\varepsilon(x_\varepsilon, y_\varepsilon) \ge \Phi_\varepsilon(x_\varepsilon, x_\varepsilon),$$

which implies

$$\frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon^2} \le v(x_\varepsilon) - v(y_\varepsilon) + \varepsilon(|x_\varepsilon|^2 - |y_\varepsilon|^2).$$

Since $v \in \mathrm{BUC}(\mathbb{R}^n)$ and $|x_\varepsilon - y_\varepsilon| \to 0$, the first term on the right tends to 0. For the second term, Step 1 gives

$$\begin{aligned}\varepsilon\big||x_\varepsilon|^2 - |y_\varepsilon|^2\big| &\le \varepsilon|x_\varepsilon - y_\varepsilon|(|x_\varepsilon| + |y_\varepsilon|) \\ &\le C\sqrt{\varepsilon}\,|x_\varepsilon - y_\varepsilon| \longrightarrow 0.\end{aligned}$$

Hence the entire right-hand side tends to 0. Therefore

$$\frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon^2} \to 0.$$

Step 3. Test the subsolution u

Set

$$M_\varepsilon = \Phi_\varepsilon(x_\varepsilon, y_\varepsilon).$$

Fix $y = y_\varepsilon$ and define the normalized upper test function

$$\varphi_\varepsilon(x) = v(y_\varepsilon) + \frac{|x - y_\varepsilon|^2}{\varepsilon^2} + \varepsilon|x|^2 + \varepsilon|y_\varepsilon|^2 + M_\varepsilon.$$

Maximality gives

$$u(x) - \varphi_\varepsilon(x) = \Phi_\varepsilon(x, y_\varepsilon) - M_\varepsilon \le 0, \qquad \varphi_\varepsilon(x_\varepsilon) = u(x_\varepsilon).$$

Thus φ_ε touches u from above at x_ε. The constant M_ε changes no derivatives. By the viscosity subsolution property,

$$u(x_\varepsilon) + H\left(x_\varepsilon, \frac{2(x_\varepsilon - y_\varepsilon)}{\varepsilon^2} + 2\varepsilon x_\varepsilon\right) \le 0.$$

Step 4. Test the supersolution v

Fix $x = x_\varepsilon$ and define the normalized lower test function

$$\psi_\varepsilon(y) = u(x_\varepsilon) - \frac{|x_\varepsilon - y|^2}{\varepsilon^2} - \varepsilon|x_\varepsilon|^2 - \varepsilon|y|^2 - M_\varepsilon.$$

Then

$$v(y) - \psi_\varepsilon(y) = M_\varepsilon - \Phi_\varepsilon(x_\varepsilon, y) \geq 0, \qquad \psi_\varepsilon(y_\varepsilon) = v(y_\varepsilon).$$

Thus ψ_ε touches v from below at y_ε. The required correction is $-M_\varepsilon$, and the viscosity supersolution property gives

$$v(y_\varepsilon) + H\left(y_\varepsilon, \frac{2(x_\varepsilon - y_\varepsilon)}{\varepsilon^2} - 2\varepsilon y_\varepsilon\right) \geq 0.$$

Step 5. Combine the two inequalities

Subtracting the supersolution inequality from the subsolution inequality yields

$$u(x_\varepsilon) - v(y_\varepsilon) \leq H\left(y_\varepsilon, \frac{2(x_\varepsilon - y_\varepsilon)}{\varepsilon^2} - 2\varepsilon y_\varepsilon\right) - H\left(x_\varepsilon, \frac{2(x_\varepsilon - y_\varepsilon)}{\varepsilon^2} + 2\varepsilon x_\varepsilon\right).$$

We estimate the right-hand side in two pieces.

First, by Lipschitz continuity in p,

$$\left|H\left(y_\varepsilon, \frac{2(x_\varepsilon - y_\varepsilon)}{\varepsilon^2} - 2\varepsilon y_\varepsilon\right) - H\left(y_\varepsilon, \frac{2(x_\varepsilon - y_\varepsilon)}{\varepsilon^2} + 2\varepsilon x_\varepsilon\right)\right| \leq 2C\varepsilon(|x_\varepsilon| + |y_\varepsilon|) \to 0.$$

Second, by Lipschitz continuity in x,

$$\left|H\left(y_\varepsilon, \frac{2(x_\varepsilon - y_\varepsilon)}{\varepsilon^2} + 2\varepsilon x_\varepsilon\right) - H\left(x_\varepsilon, \frac{2(x_\varepsilon - y_\varepsilon)}{\varepsilon^2} + 2\varepsilon x_\varepsilon\right)\right| \leq C(1 + |p_\varepsilon|)\,|x_\varepsilon - y_\varepsilon|,$$

where

$$p_\varepsilon = \frac{2(x_\varepsilon - y_\varepsilon)}{\varepsilon^2} + 2\varepsilon x_\varepsilon.$$

More explicitly,

$$\begin{aligned}(1 + |p_\varepsilon|)|x_\varepsilon - y_\varepsilon| \leq |x_\varepsilon - y_\varepsilon| + \frac{2|x_\varepsilon - y_\varepsilon|^2}{\varepsilon^2} \\ + 2\varepsilon|x_\varepsilon|\,|x_\varepsilon - y_\varepsilon| \longrightarrow 0\end{aligned}$$

by Steps 1 and 2.

Therefore

$$\limsup_{\varepsilon \to 0}(u(x_\varepsilon) - v(y_\varepsilon)) \leq 0.$$

On the other hand, since

$$\Phi_\varepsilon(x_\varepsilon, y_\varepsilon) \geq \frac{\sigma}{2}$$

and the penalty terms are nonnegative, we also have

$$u(x_\varepsilon) - v(y_\varepsilon) \geq \frac{\sigma}{2}$$

for all sufficiently small ε. This is a contradiction.

Hence $\sigma > 0$ is impossible, and we conclude that

$$u \leq v \qquad \text{on } \mathbb{R}^n.$$

Interpretation.
The doubling-of-variables method is the viscosity replacement for the classical but unavailable argument "evaluate both equations at the same maximum point of $u - v$ and subtract." The penalty forces two nearby touching points where the viscosity inequalities can be applied separately.

The diagonal penalty is easiest to see in the (x, y)-plane. As ε decreases, the ridge of the penalized objective becomes narrower around $x = y$, and its maximizing pair is forced closer to the diagonal. For each fixed $\varepsilon > 0$, the quadratic confinement term ensures that a maximum is attained. The maximizing points need not remain in a fixed bounded set as $\varepsilon \to 0$; Step 1 supplies the weaker bound needed in the proof.

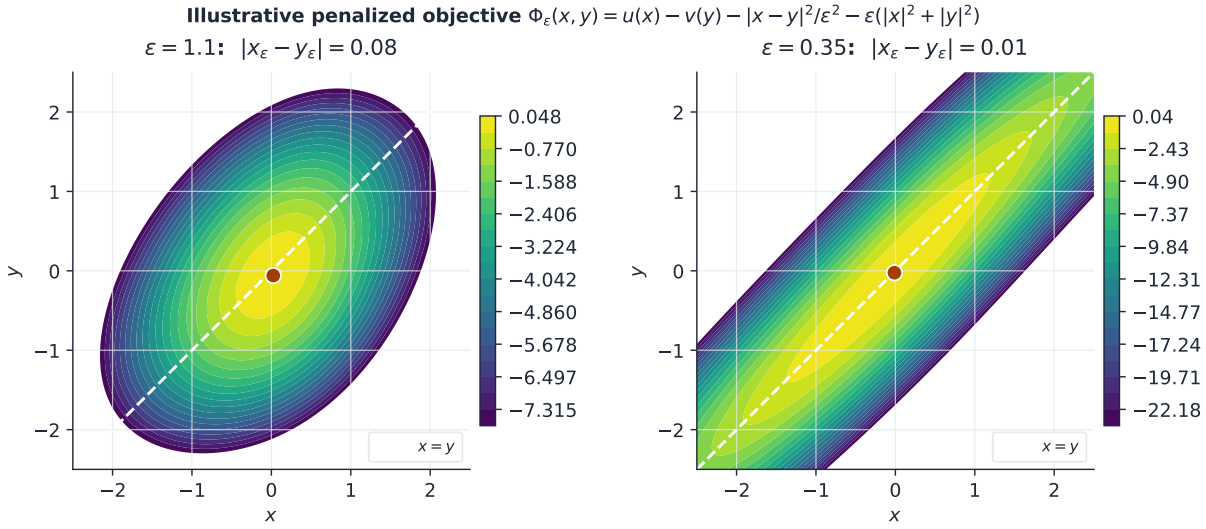


Figure 9.1. An illustrative penalized objective Φ_ε. The example functions u and v are chosen only to display the geometry of the proof. The essential feature is the concentration of maximizing pairs near $x = y$ as the diagonal penalty strengthens.

6. Uniqueness (Tran Corollary 1.20, p.19)

Under (1.13), if $u, v \in \mathrm{BUC}(\mathbb{R}^n)$ are both viscosity solutions of the static equation, then $u = v$ on $\mathbb{R}^n$.

Indeed, every solution is simultaneously a subsolution and a supersolution. Applying the comparison principle to (u, v) gives $u \le v$, and applying it to (v, u) gives $v \le u$. Therefore $u = v$.

7. Relaxing the Global Lipschitz Assumption in p (Tran Remark 1.21, p.20)

The proof above evaluates H only at gradients of touching test functions. If u and v are bounded and globally L-Lipschitz, every upper or lower test function at a contact point has gradient of magnitude at most L. This follows by comparing the touching inequality with $|u(x + h) - u(x)| \le L|h|$ in each direction, and similarly for v.

In particular, the two gradients in Steps 3 and 4,

$$p_\varepsilon^+ = \frac{2(x_\varepsilon - y_\varepsilon)}{\varepsilon^2} + 2\varepsilon x_\varepsilon, \qquad p_\varepsilon^- = \frac{2(x_\varepsilon - y_\varepsilon)}{\varepsilon^2} - 2\varepsilon y_\varepsilon,$$

lie in $\overline{B}(0, L)$, and

$$|p_\varepsilon^+ - p_\varepsilon^-| \le 2\varepsilon(|x_\varepsilon| + |y_\varepsilon|) \to 0.$$

For this bounded Lipschitz class, it is therefore enough that H be uniformly continuous on $\mathbb{R}^n \times B(0, R)$ for every $R > 0$. Choose $R > L$. Since $|x_\varepsilon - y_\varepsilon| \to 0$, uniform continuity gives

$$H(y_\varepsilon, p_\varepsilon^-) - H(x_\varepsilon, p_\varepsilon^+) \to 0,$$

which yields the same contradiction as in §5. This proves the localized comparison statement needed for Perron's construction under (1.33); global Lipschitz continuity of H is not required there.

This is important for coercive Hamiltonians such as

$$H(x,p) = \frac{1}{2}|p|^2 + V(x),$$

which, for bounded uniformly continuous V, are uniformly continuous on every bounded p-region but are not globally Lipschitz in p. Thus comparison is often proved in two stages:

1. obtain an a priori gradient estimate,
2. apply the localized comparison argument.

8. Perron's Method: The Basic Idea

Comparison gives uniqueness. We now turn to existence.

The philosophy of Perron's method is simple: collect all admissible subsolutions lying below a fixed upper barrier, take their pointwise supremum, and show that this envelope is actually a solution.

The key difficulty is that the supremum of subsolutions must remain a subsolution, and the envelope must also satisfy the supersolution condition. The first point is a stability statement; the second is a maximality argument.

9. Coercivity Assumption (Tran (1.33), p.29)

For Perron's method, assume:

$$H \in \mathrm{BUC}(\mathbb{R}^n \times B(0,R)) \qquad \text{for every } R > 0,$$

and

$$\lim_{|p|\to\infty} \inf_{x\in\mathbb{R}^n} H(x,p) = +\infty.$$

The second condition is the standard **coercivity** assumption. It means that large gradients are strongly penalized by the Hamiltonian, uniformly in x.

> **Heuristic meaning.**
> Coercivity prevents admissible solutions from developing arbitrarily large slopes. This is why Perron's construction is naturally carried out inside a bounded Lipschitz class.

10. Sup-Stability of Subsolutions (Tran Lemma 1.29, p.28)

Let $\{v_\alpha\}_{\alpha\in A}$ be a family of viscosity subsolutions of the static equation, and define

$$u(x) = \sup_{\alpha\in A} v_\alpha(x).$$

If u is finite and continuous, then u is also a viscosity subsolution. Here each v_α is assumed continuous and H is continuous.

This lemma is the technical heart of Perron's method. It says that the subsolution class is stable under taking the supremum, provided continuity is preserved.

Proof

Let $\varphi \in C^1$ touch u from above at x_0, with equality there. By adding $|x-x_0|^2$, if necessary, make the contact strict without changing the value or gradient at x_0. Choose $r > 0$ so that

$$(u-\varphi)(x) < 0 \quad \text{on } \overline{B}_r(x_0) \setminus \{x_0\}.$$

Choose members v_j of the family with $0 \le u(x_0) - v_j(x_0) \le 1/j$, and let x_j maximize $v_j - \varphi$ on $\overline{B}_r(x_0)$. Since $v_j \le u$,

$$-\frac{1}{j} \le (v_j - \varphi)(x_j) \le (u - \varphi)(x_j) \le 0.$$

Compactness and strictness imply $x_j \to x_0$, so these maxima are interior for large j, and $v_j(x_j) \to u(x_0)$. Normalize the test by

$$\varphi_j(x) = \varphi(x) + v_j(x_j) - \varphi(x_j).$$

The subsolution inequality is

$$v_j(x_j) + H(x_j, D\varphi(x_j)) \le 0.$$

Passing to the limit gives $u(x_0) + H(x_0, D\varphi(x_0)) \le 0$. Notice that we used $v_j \le u$ and approximation at x_0; local uniform convergence of the family was not assumed.

The dual statement is also valid: under the corresponding finiteness and continuity assumptions, the infimum of viscosity supersolutions is a viscosity supersolution. What generally fails is the stronger assertion that the infimum of viscosity solutions is again a solution. For convex Hamiltonians satisfying the assumptions in §9, Tran's Corollary 2.36 gives this stronger stability of Lipschitz solutions under infima, provided the infimum is finite and continuous.

11. Perron's Construction (Tran Theorems 1.30 and 1.31, pp.30–31)

Choice of the constants and the admissible class

Set

$$C_0 := \sup_{x \in \mathbb{R}^n} |H(x, 0)|,$$

which is finite by (1.33) and may equal zero. The constant functions $\underline{u} \equiv -C_0$ and $\overline{u} \equiv C_0$ satisfy

$$-C_0 + H(x, 0) \le 0, \qquad C_0 + H(x, 0) \ge 0.$$

They are a classical subsolution and supersolution, respectively. Use uniform coercivity to choose $C_1 > 0$ such that $H(x, p) > C_0 + 1$ whenever $|p| > C_1$. Equivalently,

$$H(x, p) \le C_0 + 1 \quad \Longrightarrow \quad |p| \le C_1 \qquad \text{for all } (x, p) \in \mathbb{R}^n \times \mathbb{R}^n.$$

The extra margin $+1$ will control the gradient of a local test function in the upward perturbation below. These constants are chosen from H, rather than being arbitrary positive numbers.

Consider the class

$$\mathcal{S} = \left\{ v \in \mathrm{BUC}(\mathbb{R}^n) : \begin{array}{l} -C_0 \le v \le C_0, \quad \mathrm{Lip}(v) \le C_1, \\ v \text{ is a viscosity subsolution of } v + H(x, Dv) = 0 \end{array} \right\}.$$

Thus every member is bounded and C_1-Lipschitz. For a Lipschitz function, the condition $\mathrm{Lip}(v) \le C_1$ is equivalent to the almost-everywhere gradient bound $\|Dv\|_{L^\infty} \le C_1$ used in Tran (1.34). Define

$$u(x) = \sup_{v \in \mathcal{S}} v(x).$$

Theorem 1.30. Under (1.33), this envelope u is a Lipschitz viscosity solution of the static equation.

In the notation of Tran (1.34),

$$u(x) = \sup \left\{ v(x) : \begin{array}{l} -C_0 \le v \le C_0, \quad \|Dv\|_{L^\infty} \le C_1, \\ v \text{ is a Lipschitz viscosity subsolution} \end{array} \right\}.$$

Step 1. The envelope is an admissible subsolution

The class is nonempty because $-C_0 \in \mathcal{S}$. The envelope obeys $-C_0 \le u \le C_0$, and, for every $v \in \mathcal{S}$,

$$v(x) \le v(y) + C_1|x-y| \le u(y) + C_1|x-y|.$$

Taking the supremum and exchanging x, y shows that u is C_1-Lipschitz. Lemma 1.29 then shows that u is a viscosity subsolution. In particular, $u \in \mathcal{S}$.

Step 2. A violating lower test cannot meet the upper barrier

Suppose, toward a contradiction, that a lower test function $\psi \in C^1$ touches u at x_0, with

$$\psi(x_0) = u(x_0), \qquad \psi(x_0) + H(x_0, D\psi(x_0)) < 0.$$

Replacing ψ by $\psi - |x-x_0|^2$ if needed, we may assume $u > \psi$ locally away from x_0, without changing the displayed inequality.

If $u(x_0) = C_0$, then $\psi \le u \le C_0$ near x_0, with equality there. Hence ψ also touches the constant supersolution C_0 from below. Its supersolution inequality gives

$$\psi(x_0) + H(x_0, D\psi(x_0)) \ge 0,$$

a contradiction. Thus $u(x_0) < C_0$.

Step 3. Choose a local upward perturbation

By continuity, choose $r > 0$ and $\eta > 0$ sufficiently small so that, on $\overline{B}_r(x_0)$,

$$u \le C_0 - 2\eta, \qquad \psi \ge -C_0 - 1, \qquad \psi + H(x, D\psi) \le -2\eta.$$

The radius is also chosen inside the neighborhood where $\psi \le u$, with strict inequality away from x_0. Consequently,

$$H(x, D\psi) \le -\psi - 2\eta \le C_0 + 1, \qquad |D\psi| \le C_1.$$

This is where the choice of C_1 is used. Strict contact and compactness give

$$\gamma = \min_{\partial B_r(x_0)} (u - \psi) > 0.$$

Choose $0 < \delta < \min\{\eta, \gamma/2\}$. Then $\psi + \delta < u$ on the boundary and in a neighborhood of it. On the ball,

$$(\psi + \delta) + H(x, D(\psi + \delta)) \le -2\eta + \delta < 0,$$

so $\psi + \delta$ is a classical strict subsolution. Also,

$$\psi + \delta \le u + \delta \le C_0 - 2\eta + \delta < C_0.$$

Step 4. Patch and contradict maximality

Define

$$\widehat{u}(x) = \begin{cases} \max\{u(x), \psi(x) + \delta\}, & x \in B_r(x_0), \\ u(x), & x \notin B_r(x_0). \end{cases}$$

We check every constraint in $\mathcal{S}$:

- **Bounds:** $-C_0 \le u \le \widehat{u} \le C_0$.
- **Subsolution property:** inside the ball, the maximum of the two subsolutions is a subsolution by Lemma 1.29. Near the boundary, $\widehat{u} = u$, so the patch introduces no new contact issue.

- **Lipschitz bound:** both u and $\psi + \delta$ are C_1-Lipschitz on the ball, so their maximum is too. Outside, $\hat{u} = u$. For a segment crossing the boundary, split it at its entry and exit points and add these estimates; this gives the same global constant C_1.

Thus $\hat{u} \in \mathcal{S}$. However,

$$\hat{u}(x_0) = u(x_0) + \delta > u(x_0),$$

contradicting the definition of u as the supremum over $\mathcal{S}$. The envelope is therefore also a supersolution, proving Theorem 1.30.

Uniqueness: Theorem 1.31

Under (1.33), the function constructed above is the unique Lipschitz viscosity solution. For bounded Lipschitz solutions, this follows by applying the localized comparison argument in §7 in both directions. The uniform continuity assumption in (1.33) is sufficient; the global estimates (1.13) are not additional assumptions here.

In fact, any globally Lipschitz viscosity solution is bounded under (1.33). It is differentiable almost everywhere, and the stationary version of Week 8's consistency argument gives $u(x) = -H(x, Du(x))$ there. Boundedness of H on the region containing these gradients, followed by continuity of u, gives a bound everywhere. Thus the bounded Lipschitz comparison covers the stated class.

12. Why Perron's Method Is Conceptually Important

In optimal control, one often first obtains a candidate solution from a value function or from a representation formula. Perron's method shows that viscosity solutions can also be constructed **purely analytically**, without first solving the control problem explicitly.

Thus two different routes meet at the same object:

- the **control route** produces a candidate through optimization,
- the **PDE route** produces a candidate through the maximal subsolution envelope.

Comparison then guarantees that both constructions agree for the same equation and in the same comparison class.

13. Example 9.1: The Eikonal Equation in $B_1(0)$

Consider

$$|Du| = 1 \qquad \text{in } B_1(0), \qquad u = 0 \qquad \text{on } \partial B_1(0).$$

The unique viscosity solution is

$$u(x) = \operatorname{dist}(x, \partial B_1(0)) = 1 - |x|.$$

- **Away from the origin**, u is smooth and

$$Du(x) = -\frac{x}{|x|} \qquad (x \neq 0),$$

so

$$|Du(x)| = 1.$$

Hence the equation holds classically in $B_1(0) \setminus \{0\}$.

- **At the origin**, the function is not differentiable. However,

$$D^+u(0) = \overline{B(0,1)}, \qquad D^-u(0) = \emptyset.$$

Therefore the subsolution condition for the equation

$$|Du| - 1 = 0$$

becomes

$$|p| - 1 \le 0 \qquad \text{for all } p \in D^+u(0),$$

which is true because $|p| \le 1$ on $\overline{B(0,1)}$.
The supersolution condition at 0 is vacuous since $D^-u(0) = \emptyset$.

Hence $u(x) = 1 - |x|$ is a viscosity solution.

Why the Dirichlet solution is unique

This equation has no zeroth-order term, so Theorem 1.18 does not apply directly. A short strict-subsolution argument proves uniqueness for this Dirichlet problem. Let $a, b \in C(\overline{B_1(0)})$ be viscosity solutions with zero boundary values, and fix $0 < \theta < 1$. Then θa satisfies the strict subsolution bound $|D(\theta a)| \le \theta < 1$ in the viscosity sense.

If $\max(\theta a - b) > 0$, maximize

$$\theta a(x) - b(y) - \frac{|x-y|^2}{\varepsilon^2}$$

on $\overline{B_1(0)} \times \overline{B_1(0)}$. Compactness, continuity, and the zero boundary values imply that, for small ε, a maximizing pair lies in the interior: any limiting pair is diagonal and realizes a positive maximum of $\theta a - b$. The normalized upper and lower tests, as in §5, have the same gradient $p_\varepsilon = 2(x_\varepsilon - y_\varepsilon)/\varepsilon^2$. They give

$$|p_\varepsilon| \le \theta \qquad \text{and} \qquad |p_\varepsilon| \ge 1,$$

a contradiction. Thus $\theta a \le b$. Let $\theta \uparrow 1$ and exchange a, b to obtain $a = b$.

> **Geometric interpretation.**
> The solution is the distance from x to the boundary. The level sets are concentric spheres, and the equation $|Du| = 1$ expresses that the function increases at unit rate in the inward normal direction. This is a model example showing that viscosity solutions correctly capture geometrically meaningful nonsmooth solutions.

14. Summary

This week establishes the basic well-posedness mechanism for static Hamilton–Jacobi equations.

- Under the Lipschitz assumptions (1.13), the **comparison principle** implies

$$\text{subsolution} \le \text{supersolution}.$$

- As a consequence, viscosity solutions are **unique**.
- Under the coercivity assumption (1.33), **Perron's method** constructs a viscosity solution as the supremum of bounded Lipschitz subsolutions.
- Theorem 1.31 and comparison identify the constructed solution as the unique Lipschitz solution.
- The eikonal problem in the unit ball provides a canonical example in which the unique viscosity solution is the distance to the boundary.

These two results — comparison and Perron — are the foundational existence/uniqueness tools for the theory developed in the next weeks.

15. Exercises

Exercise 9.1

Fill in the details of Step 2 in the doubling-of-variables proof, and show carefully that

$$\frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon^2} \to 0.$$

Exercise 9.2

Identify precisely where the two assumptions in (1.13) are used in Step 5 of the comparison proof.

Exercise 9.3

Explain why the maximum of two viscosity subsolutions is again a viscosity subsolution.

Exercise 9.4

Verify directly, including the nonsmooth point $x = 0$, that

$$u(x) = 1 - |x|$$

is a viscosity solution of

$$|Du| - 1 = 0 \quad \text{in } B_1(0), \qquad u = 0 \quad \text{on } \partial B_1(0).$$

Compute $D^+u(0)$ and $D^-u(0)$, and check both viscosity inequalities.

Exercise 9.5

Compare the present Perron construction with the vanishing viscosity method from Week 8. What is the main conceptual difference between the two existence proofs?

Exercise 9.6

Prove that the minimum of two viscosity supersolutions of a continuous first-order Hamilton–Jacobi equation is again a viscosity supersolution. Explain why this result alone does not imply that the minimum of two viscosity solutions is again a solution.

16. References

- H. V. Tran, *Hamilton–Jacobi Equations: Theory and Applications*, AMS Graduate Studies in Mathematics 213, 2021.