Week 10. Cauchy Comparison, Lipschitz Regularity, and Finite Propagation

Tran: §1.6 (Theorem 1.23), §1.9 (Theorems 1.36, 1.38), §1.10 (Theorem 1.39)

1. Overview

In this lecture we study three basic properties of viscosity solutions to the Cauchy problem

$$\begin{cases} u_t + H(x, Du) = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ u(x, 0) = u_0(x) & \text{on } \mathbb{R}^n. \end{cases} \tag{C}$$

2. Learning Objectives

By the end of this lecture, students should be able to:

1. prove the **comparison principle** for the Cauchy problem and hence uniqueness of viscosity solutions;
2. derive the L^∞**-contraction property** and use it to obtain **time Lipschitz continuity**;
3. combine time regularity with coercivity of the Hamiltonian in order to obtain **space Lipschitz continuity**;
4. prove the **finite speed of propagation** property and interpret it as a domain-of-dependence statement.

> **Position in the course.**
> In this course we introduced Hamilton–Jacobi equations through **optimal control and dynamic programming** before studying viscosity theory in full generality. Therefore the results of this week should be viewed not only as abstract PDE theorems, but also as the rigorous well-posedness statements that justify the value-function interpretation.

3. Standing Assumptions

As in Weeks 8–9, D acts on the spatial variable, φ denotes an upper test function, and ψ denotes a lower test function. Contact values are matched by adding constants, which do not change derivatives. Throughout, subsolutions and supersolutions are understood to belong to $\mathrm{BUC}(\mathbb{R}^n \times [0, T])$ on each finite time strip under consideration; boundedness for all $t \geq 0$ at once is not required.

For the comparison principle we assume the structural condition

$$|H(x, p) - H(x, q)| \leq C|p - q|, \qquad |H(x, p) - H(y, p)| \leq C(1 + |p|)\,|x - y| \tag{1.13}$$

for all $x, y, p, q \in \mathbb{R}^n$, for some constant $C > 0$.

Connection with optimal control. In Weeks 2–4, we used

$$H(x, p) = \sup_{v \in V} \{-b(x, v) \cdot p - f(x, v)\},$$

with $|b(x, v)| \leq M_b$ and Lipschitz constants L_b, L_f in x, uniformly in v. Comparing the expressions inside the supremum gives

$$\begin{aligned} |H(x, p) - H(x, q)| &\leq M_b|p - q|, \\ |H(x, p) - H(y, p)| &\leq (L_b|p| + L_f)|x - y|. \end{aligned}$$

Thus (1.13) follows automatically from our control assumptions, with $C = \max\{1, M_b, L_b, L_f\}$. Coercivity, however, is an additional assumption and does not follow from these bounds alone.

For Lipschitz regularity we additionally assume the coercivity condition

$$H \in \mathrm{BUC}(\mathbb{R}^n \times B(0, R)) \text{ for each } R > 0, \qquad \lim_{|p| \to \infty} \inf_{x \in \mathbb{R}^n} H(x, p) = +\infty, \tag{1.33}$$

and the regularity of the initial data

$$u_0 \in C^1(\mathbb{R}^n), \qquad \|u_0\|_{C^1(\mathbb{R}^n)} < \infty. \tag{1.36}$$

4. Extrema at the Terminal Time

Before proving the comparison principle for the Cauchy problem, we need a boundary-in-time lemma.

Lemma 10.1 (Tran Lemma 1.22).
Fix $T > 0$. Let $u \in C(\mathbb{R}^n \times [0,T])$ be a viscosity subsolution of

$$u_t + H(x, Du) = 0 \quad \text{in } \mathbb{R}^n \times (0,T).$$

Assume that $\varphi \in C^1(\mathbb{R}^n \times [0,T])$ and that $u - \varphi$ has a strict maximum at (x_0, T) over $\mathbb{R}^n \times (0,T]$. Then

$$\varphi_t(x_0,T) + H(x_0, D\varphi(x_0,T)) \le 0.$$

Proof

The difficulty is that (x_0, T) is not an interior point in time, so we cannot directly apply the usual viscosity subsolution test. To move the touching point into the interior, define

$$\varphi_\varepsilon(x,t) = \varphi(x,t) + \frac{\varepsilon}{T-t}.$$

Work on a small closed cylinder $\overline{B}_r(x_0) \times [T-r, T]$, with $0 < r < T$. Strictness gives a positive gap on its lateral and bottom boundaries. The added term blows up as $t \uparrow T$; comparing with $(x_0, T - \sqrt{\varepsilon})$ shows that, for small ε, $u - \varphi_\varepsilon$ has an interior maximum $(x_\varepsilon, t_\varepsilon)$ and that $(x_\varepsilon, t_\varepsilon) \to (x_0, T)$.

Applying the viscosity subsolution inequality at $(x_\varepsilon, t_\varepsilon)$, we obtain

$$(\varphi_\varepsilon)_t(x_\varepsilon, t_\varepsilon) + H(x_\varepsilon, D\varphi_\varepsilon(x_\varepsilon, t_\varepsilon)) \le 0.$$

Now

$$(\varphi_\varepsilon)_t = \varphi_t + \frac{\varepsilon}{(T-t)^2}, \qquad D\varphi_\varepsilon = D\varphi.$$

Hence

$$\varphi_t(x_\varepsilon, t_\varepsilon) + \frac{\varepsilon}{(T-t_\varepsilon)^2} + H(x_\varepsilon, D\varphi(x_\varepsilon, t_\varepsilon)) \le 0.$$

Dropping the nonnegative middle term and passing to the limit $\varepsilon \to 0$, we conclude that

$$\varphi_t(x_0,T) + H(x_0, D\varphi(x_0,T)) \le 0. \qquad \square$$

> **Remark.**
> This lemma allows viscosity testing at a terminal-time contact point. The supersolution version follows by using $\psi_\varepsilon = \psi - \varepsilon/(T-t)$. Non-strict contacts are also allowed: add $|x-x_0|^2 + (t-T)^2$ to an upper test, or subtract it from a lower test, without changing the derivatives at the contact point.

5. Comparison Principle for the Cauchy Problem

We now state the main uniqueness theorem.

Theorem 10.2 (Comparison principle for the Cauchy problem; Tran Theorem 1.23).
Assume (1.13), and fix $T > 0$. Let $u, v \in \mathrm{BUC}(\mathbb{R}^n \times [0,T])$ be a viscosity subsolution and a viscosity supersolution of

$$u_t + H(x, Du) = 0 \quad \text{in } \mathbb{R}^n \times (0,T),$$

respectively. Assume also that

$$u(x,0) \le v(x,0) \qquad \text{for all } x \in \mathbb{R}^n.$$

Then

$$u(x,t) \le v(x,t) \qquad \text{for all } (x,t) \in \mathbb{R}^n \times [0,T].$$

Proof

Assume by contradiction that

$$\sup_{\mathbb{R}^n \times [0,T]} (u - v) = \sigma > 0.$$

Choose $(x_1, t_1) \in \mathbb{R}^n \times [0,T]$ such that

$$u(x_1,t_1) - v(x_1,t_1) > \frac{3\sigma}{4}.$$

Since $u(\cdot,0) \le v(\cdot,0)$, necessarily $t_1 > 0$.
Choose $\delta > 0$ sufficiently small and then choose $\varepsilon > 0$ sufficiently small so that

$$2\varepsilon|x_1|^2 + 2\delta t_1 < \frac{\sigma}{4}.$$

With δ fixed, define

$$\Phi(x,y,t,s) = u(x,t) - v(y,s) - \frac{|x-y|^2 + (t-s)^2}{\varepsilon^2} - \varepsilon(|x|^2 + |y|^2) - \delta(t+s).$$

Let $(x_\varepsilon, y_\varepsilon, t_\varepsilon, s_\varepsilon)$ be a maximum point of Φ. Then

$$\Phi(x_\varepsilon, y_\varepsilon, t_\varepsilon, s_\varepsilon) \ge \Phi(x_1, x_1, t_1, t_1) > \frac{\sigma}{2}.$$

The proof proceeds in six steps.

Step 1. Localization

For each fixed $\varepsilon > 0$, boundedness and the quadratic confinement ensure that the maximum is attained. As in Week 9, comparison with a fixed diagonal point gives, for a constant K independent of ε,

$$|x_\varepsilon - y_\varepsilon| + |t_\varepsilon - s_\varepsilon| \le K\varepsilon, \qquad |x_\varepsilon| + |y_\varepsilon| \le \frac{K}{\sqrt{\varepsilon}}.$$

Step 2. Vanishing of the penalization gap

Comparing the maximum with $\Phi(x_\varepsilon, x_\varepsilon, t_\varepsilon, t_\varepsilon)$ yields

$$\begin{aligned}\frac{|x_\varepsilon - y_\varepsilon|^2 + (t_\varepsilon - s_\varepsilon)^2}{\varepsilon^2} &\le v(x_\varepsilon,t_\varepsilon) - v(y_\varepsilon,s_\varepsilon) \\ &\quad + \varepsilon(|x_\varepsilon|^2 - |y_\varepsilon|^2) + \delta(t_\varepsilon - s_\varepsilon) \longrightarrow 0.\end{aligned}$$

The first term tends to zero by uniform continuity. For the second, use $\varepsilon\big||x_\varepsilon|^2 - |y_\varepsilon|^2\big| \le \varepsilon|x_\varepsilon - y_\varepsilon|(|x_\varepsilon| + |y_\varepsilon|)$ and Step 1. The last term also tends to zero, with δ fixed.

Step 3. Separation from the initial time

This is the genuinely new step in the Cauchy problem. One shows that there exists $\mu > 0$ such that, for all sufficiently small ε,

$$t_\varepsilon \ge \mu, \qquad s_\varepsilon \ge \mu.$$

Indeed, if $t_\varepsilon \to 0$ or $s_\varepsilon \to 0$, then by the uniform continuity of u and v and the initial condition $u(\cdot,0) \le v(\cdot,0)$, the maximal value of Φ cannot remain positive, contradicting the choice of $\sigma > 0$.

Step 4. Subsolution test

Set $M_\varepsilon = \max \Phi$ and fix $(y_\varepsilon, s_\varepsilon)$. The function $u - \varphi$ has maximum zero at $(x_\varepsilon, t_\varepsilon)$, where

$$\varphi(x,t) = v(y_\varepsilon, s_\varepsilon) + \frac{|x-y_\varepsilon|^2 + (t-s_\varepsilon)^2}{\varepsilon^2} + \varepsilon(|x|^2 + |y_\varepsilon|^2) + \delta(t+s_\varepsilon) + M_\varepsilon.$$

Thus $u = \varphi$ at the contact point. If $t_\varepsilon = T$, use Lemma 10.1; otherwise use the interior test. The subsolution inequality gives

$$\frac{2(t_\varepsilon - s_\varepsilon)}{\varepsilon^2} + \delta + H\left(x_\varepsilon, \frac{2(x_\varepsilon - y_\varepsilon)}{\varepsilon^2} + 2\varepsilon x_\varepsilon\right) \le 0.$$

Step 5. Supersolution test

Fix $(x_\varepsilon, t_\varepsilon)$ and define the lower test

$$\begin{aligned}\psi(y,s) = u(x_\varepsilon, t_\varepsilon) &- \frac{|x_\varepsilon - y|^2 + (t_\varepsilon - s)^2}{\varepsilon^2} \\ &- \varepsilon(|x_\varepsilon|^2 + |y|^2) - \delta(t_\varepsilon + s) - M_\varepsilon.\end{aligned}$$

Then $v - \psi$ has minimum zero at $(y_\varepsilon, s_\varepsilon)$. Using the supersolution version of Lemma 10.1 if $s_\varepsilon = T$, we obtain

$$\frac{2(t_\varepsilon - s_\varepsilon)}{\varepsilon^2} - \delta + H\left(y_\varepsilon, \frac{2(x_\varepsilon - y_\varepsilon)}{\varepsilon^2} - 2\varepsilon y_\varepsilon\right) \ge 0.$$

Step 6. Subtract and pass to the limit

Subtracting the second inequality from the first, we obtain

$$2\delta \le H\left(y_\varepsilon, \frac{2(x_\varepsilon - y_\varepsilon)}{\varepsilon^2} - 2\varepsilon y_\varepsilon\right) - H\left(x_\varepsilon, \frac{2(x_\varepsilon - y_\varepsilon)}{\varepsilon^2} + 2\varepsilon x_\varepsilon\right).$$

Keeping δ fixed and using assumption (1.13), together with Step 2 and the bounds from Step 1, the right-hand side tends to 0 as $\varepsilon \to 0$. Hence

$$2\delta \le 0,$$

a contradiction. Therefore

$$u \le v \quad \text{on } \mathbb{R}^n \times [0,T]. \qquad \square$$

Corollary 10.3 (Uniqueness).
Under assumption (1.13), the Cauchy problem (C) has at most one viscosity solution in $\mathrm{BUC}(\mathbb{R}^n \times [0,T])$.

6. L^∞-Contraction

The comparison principle immediately implies stability with respect to initial data.

Proposition 10.4 (L^∞-contraction; Tran Exercise 11).
Assume (1.13). Let u_1, u_2 be viscosity solutions of (C) with initial data g_1, g_2, respectively. Then for every $t \ge 0$,

$$\sup_{x \in \mathbb{R}^n} |u_1(x,t) - u_2(x,t)| \le \sup_{x \in \mathbb{R}^n} |g_1(x) - g_2(x)|.$$

Proof

Let

$$M := \sup_{x\in\mathbb{R}^n} (g_1(x) - g_2(x)).$$

Then

$$g_1(x) \le g_2(x) + M \qquad \text{for all } x \in \mathbb{R}^n.$$

Since adding a constant does not change the equation, the function $u_2 + M$ is also a viscosity solution. By Theorem 10.2,

$$u_1(x,t) \le u_2(x,t) + M \qquad \text{for all } (x,t).$$

Hence

$$u_1(x,t) - u_2(x,t) \le \sup_x (g_1 - g_2).$$

Interchanging the roles of u_1 and u_2, we also obtain

$$u_2(x,t) - u_1(x,t) \le \sup_x (g_2 - g_1).$$

Therefore

$$\sup_x |u_1(x,t) - u_2(x,t)| \le \sup_x |g_1(x) - g_2(x)|. \qquad \square$$

Interpretation.
The solution operator for the Cauchy problem is nonexpansive in the L^∞-norm.

7. Lipschitz Regularity

We now prove that, under suitable assumptions, the solution is Lipschitz continuous in both time and space.

Theorem 10.5 (Tran Theorem 1.36).
Assume (1.13), (1.33), and (1.36). Then the Cauchy problem (C) has a unique viscosity solution u, and u is Lipschitz continuous in both space and time.

In particular, there exists $C > 0$ such that

$$|u_t(x,t)| + |Du(x,t)| \le C \quad \text{for a.e. } (x,t) \in \mathbb{R}^n \times [0,\infty). \tag{1.39}$$

Proof

For existence, we use the Cauchy version of Perron's method (Tran Theorem 1.34 and Remark 1.35), extending the static construction from Week 9. Its barriers are the functions in Step 1 below. Comparison gives uniqueness. We establish the Lipschitz estimate in three steps.

Step 1. Time Lipschitz continuity at $t = 0$

By (1.33) and (1.36), we may choose

$$C_0 \ge \sup_{x\in\mathbb{R}^n} |H(x, Du_0(x))|.$$

Then

$$u_0(x) - C_0 t \qquad \text{and} \qquad u_0(x) + C_0 t$$

are respectively a viscosity subsolution and a viscosity supersolution of (C). By comparison,

$$u_0(x) - C_0 t \le u(x,t) \le u_0(x) + C_0 t \qquad \text{for all } (x,t) \in \mathbb{R}^n \times [0,\infty).$$

Therefore

$$|u(x,t) - u_0(x)| \le C_0 t.$$

Thus u is Lipschitz in time at the initial time.

Step 2. Time Lipschitz continuity for all $t \ge 0$

Fix $s > 0$, and define

$$v(x,t) := u(x, t+s).$$

Since the Hamiltonian is independent of time, v is also a viscosity solution of the same equation, now with initial data

$$v(x,0) = u(x,s).$$

Applying Proposition 10.4 to u and v, we get

$$\|u(\cdot, t+s) - u(\cdot,t)\|_\infty \le \|u(\cdot,s) - u(\cdot,0)\|_\infty.$$

Using Step 1 on the right-hand side,

$$\|u(\cdot,s) - u(\cdot,0)\|_\infty \le C_0 s.$$

Hence

$$\|u(\cdot,t+s) - u(\cdot,t)\|_\infty \le C_0 s \qquad \text{for all } t, s \ge 0.$$

Therefore u is globally Lipschitz continuous in time.

Step 3. Space Lipschitz continuity

Fix $t_0 > 0$, and let φ touch $u(\cdot, t_0)$ from above at x_0. Make this spatial contact strict by adding $|x - x_0|^2$. On a small closed cylinder around (x_0, t_0), maximize

$$u(x,t) - \varphi(x) - \frac{(t-t_0)^2}{2\varepsilon}.$$

As in the stability-of-extrema argument from Week 8, the maximizing points $(x_\varepsilon, t_\varepsilon)$ are interior for small ε and converge to (x_0, t_0). Time Lipschitz continuity implies that the time derivative of this upper test satisfies

$$|a_\varepsilon| \le C_0, \qquad a_\varepsilon := \frac{t_\varepsilon - t_0}{\varepsilon}.$$

Indeed, compare the touching inequality in the time direction with the time Lipschitz bound. The subsolution test, normalized by a constant if needed, now gives

$$a_\varepsilon + H(x_\varepsilon, D\varphi(x_\varepsilon)) \le 0, \qquad H(x_0, D\varphi(x_0)) \le C_0$$

after passing to the limit. By uniform coercivity, choose $L > 0$ such that $H(x,p) \le C_0$ implies $|p| \le L$. Thus every spatial upper test has gradient bounded by L.

To obtain the Lipschitz estimate itself, fix y and $K > L$. If

$$u(x,t_0) - u(y,t_0) - K|x-y|$$

has a positive value, its maximum is attained at some $x \ne y$, since the time slice is bounded. The cone $u(y,t_0) + K|x-y|$, shifted by a constant, is then a smooth upper test near that point with gradient magnitude $K > L$, a contradiction. Exchanging x, y and letting $K \downarrow L$ yields

$$|u(x,t_0) - u(y,t_0)| \le L|x-y|.$$

Continuity extends this estimate to $t_0 = 0$. Together with Step 2, it gives the claimed almost-everywhere derivative bounds without assuming spatial differentiability in advance.

This proves the theorem. □

8. Global Lipschitz Well-Posedness

The following theorem strengthens Theorem 10.5 by removing the global structural assumption (1.13).

Theorem 10.6 (Tran Theorem 1.38).
Assume (1.33) and (1.36). Then the Cauchy problem (C) has a unique viscosity solution u, and u is Lipschitz continuous in both space and time. In particular, there exists $C > 0$ such that

$$|Du(x,t)| + |u_t(x,t)| \le C \quad \text{for a.e. } (x,t) \in \mathbb{R}^n \times [0,\infty). \tag{1.40}$$

To obtain this strengthening, choose C_0 as in §7 and then $R > \max\{\|Du_0\|_\infty, C_0+1\}$ so large that $H(x,p) > C_0+1$ whenever $|p| > R$. Modify H outside $|p| \le R$, using a smooth cutoff to interpolate to $|p|$ for $|p| \ge 2R$. The modified Hamiltonian $\widetilde{H}$ is globally uniformly continuous and coercive, and satisfies $\widetilde{H} > C_0$ for $|p| > R$.

Global uniform continuity makes the Hamiltonian difference in Step 6 of §5 tend to zero, so comparison and the preceding Perron and Lipschitz arguments apply to the modified problem. The time bound is still C_0, and the spatial upper-test estimate forces the solution to be R-Lipschitz in space. All upper and lower test gradients therefore lie in $|p| \le R$, where $\widetilde{H} = H$, so this solution also solves the original problem.

For uniqueness, compare this Lipschitz solution with any other solution in the finite-strip class of §3. In the doubling argument, the test gradient on the Lipschitz side is bounded; the other gradient differs from it by a quantity tending to zero. Uniform continuity of H on bounded p-regions then gives the same contradiction. This is the time-dependent version of Tran's Remark 1.21.

9. Finite Speed of Propagation

We now turn to the final main theorem of this lecture.

Theorem 10.7 (Finite speed of propagation; Tran Theorem 1.39).
Assume (1.13). Let u be a viscosity subsolution and v a viscosity supersolution of

$$u_t + H(x, Du) = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty).$$

Assume further that

$$u(x,0) \le v(x,0) \qquad \text{for all } x \in B(0,R)$$

for some $R > 0$. Then, with the constant $C > 0$ from (1.13),

$$u(x,t) \le v(x,t) \qquad \text{for all } x \in B(0, R - Ct), \quad 0 \le t \le R/C.$$

Interpretation.
This theorem says that local information in the initial data propagates at finite speed. If two solutions are initially ordered on a ball $B(0,R)$, then the ordering persists for positive times inside the shrinking ball $B(0, R-Ct)$.

In other words, the solution at (x,t) depends only on the initial data in a suitable domain of dependence.

In one space dimension, the theorem produces a literal space-time cone. The interval on which the initial ordering is known shrinks inward at speed C. Points inside the shaded region are still protected from information originating outside $B(0,R)$.

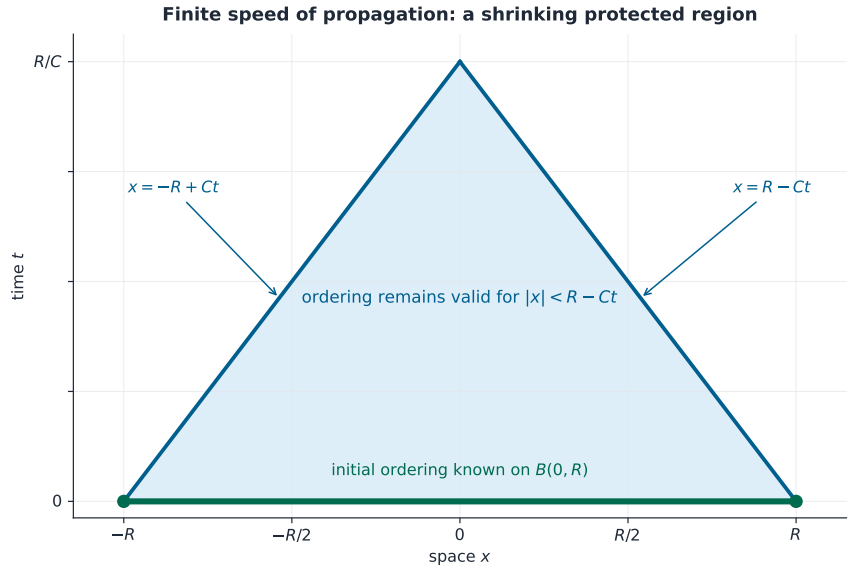


Figure 10.1. If $u(\cdot,0) \le v(\cdot,0)$ on $B(0,R)$, then $u(\cdot,t) \le v(\cdot,t)$ on the shrinking ball $B(0,R-Ct)$. The boundary lines have slopes determined by the propagation speed C.

10. Proof Structure of Finite Speed of Propagation

Let

$$w := u - v.$$

The proof begins with a reduction to a simpler first-order inequality.

Lemma 10.8 (Tran Lemma 1.40).
Under assumption (1.13), the function $w = u - v$ is a viscosity subsolution of

$$w_t - C|Dw| = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty). \tag{1.42}$$

Equivalently, $w_t - C|Dw| \le 0$ in the viscosity sense. Thus the nonlinear comparison problem is reduced to an eikonal-type equation.

Proof

Let φ touch w strictly from above at an interior point (x_0,t_0). Localize to a small closed cylinder and maximize

$$u(x,t) - v(y,s) - \varphi(x,t) - \frac{|x-y|^2 + (t-s)^2}{\varepsilon^2}.$$

The same estimates as in §5 give convergence of the maximizing points to (x_0,x_0,t_0,t_0) and vanishing of the penalty quotient. Subtracting the two viscosity inequalities and using (1.13) gives

$$\varphi_t(x_\varepsilon,t_\varepsilon) - C|D\varphi(x_\varepsilon,t_\varepsilon)| \le C|x_\varepsilon - y_\varepsilon| + \frac{2C|x_\varepsilon - y_\varepsilon|^2}{\varepsilon^2} \longrightarrow 0.$$

Passing to the limit proves the lemma. □

The next ingredient is an explicit barrier.

Lemma 10.9 (Tran Lemma 1.41).
If w is a viscosity subsolution of (1.42) and $w(x,0) \le 0$ on $B(0,R)$, then $w(x,t) \le 0$ whenever $|x| + Ct < R$.

Proof

Set $M = \max\{0, \sup_x w(x,0)\}$. For $0 < \varepsilon < R$, choose a smooth nondecreasing cutoff ξ_ε with $0 \le \xi_\varepsilon \le M$, such that

$$\xi_\varepsilon(s) = 0 \quad (s \le R - \varepsilon), \qquad \xi_\varepsilon(s) = M \quad (s \ge R).$$

Define

$$\psi_\varepsilon(x,t) = \xi_\varepsilon(|x| + Ct), \qquad 0 \le t < T_\varepsilon := \frac{R-\varepsilon}{C}.$$

For these times the function vanishes near $x = 0$, so it is smooth there despite the appearance of $|x|$. For $x \neq 0$,

$$(\psi_\varepsilon)_t - C|D\psi_\varepsilon| = C\xi_\varepsilon' - C|\xi_\varepsilon'| = 0.$$

Also $w(x,0) \le \psi_\varepsilon(x,0)$ everywhere: inside the ball use $w(x,0) \le 0$, and outside use $w(x,0) \le M$. Apply Theorem 10.2 to the reduced Hamiltonian $-C|p|$ on each strip $[0,T]$ with $T < T_\varepsilon$. This gives $w \le \psi_\varepsilon$.

If $|x| + Ct < R$, choose $0 < \varepsilon < R - |x| - Ct$. Then $t < T_\varepsilon$ and $\psi_\varepsilon(x,t) = 0$, so $w(x,t) \le 0$. □

Applying this lemma to $w = u - v$ proves Theorem 10.7.

> **Geometric meaning.**
> The set $B(0, R - Ct)$ is the interior region that still remains protected by the initial ordering at time t. This is the domain-of-dependence picture for the Hamilton–Jacobi flow.

11. Example 10.1: A Burgers-Type One-Dimensional Equation

Consider

$$u_t + \frac{1}{2}(u_x)^2 = 0 \qquad \text{in } \mathbb{R} \times (0,\infty), \qquad u(x,0) = u_0(x).$$

This corresponds to the convex Hamiltonian

$$H(p) = \frac{1}{2}p^2.$$

Observation 1.
The Hamiltonian $H(p) = \frac{1}{2}p^2$ is convex and coercive, but it is **not globally Lipschitz in** p. Therefore assumption (1.13) is not satisfied globally.

Observation 2.
Nevertheless, suppose that the subsolution and supersolution being compared have a common a priori Lipschitz bound $|u_x|, |v_x| \le R$. One may then modify H outside $[-R, R]$ without changing either viscosity inequality. On $[-R, R]$, the Hamiltonian is Lipschitz with constant R, so localized comparison and finite propagation apply in this Lipschitz class, with propagation speed $C = R$ (or any larger Lipschitz constant for the modified Hamiltonian).

Observation 3.
This example prepares for the Hopf–Lax formula, which will be studied later for convex Hamiltonians. For the present week, the point is not to solve the equation explicitly, but to understand that comparison, regularity, and propagation estimates already yield important qualitative information.

> **Preview.**
> If one formally differentiates the equation and sets $w = u_x$, then w satisfies the inviscid Burgers equation
>
> $$w_t + ww_x = 0.$$
>
> This explains heuristically why bounded slopes correspond to bounded propagation speed.

12. Summary

In this lecture we established the basic well-posedness theory for the Cauchy problem for first-order Hamilton–Jacobi equations.

1. The **terminal-time extrema lemma** allows viscosity testing at $t = T$.
2. The **comparison principle** gives uniqueness for the Cauchy problem.
3. The **L^∞-contraction estimate** yields stability with respect to initial data.

4. **Time Lipschitz continuity** follows from a barrier argument and time translation.
5. **Space Lipschitz continuity** follows from coercivity once time regularity is known.
6. The **finite speed of propagation theorem** gives a precise domain-of-dependence statement.

These results form the foundation for the study of convex Hamiltonians and representation formulas in the following weeks.

13. Exercises

Exercise 10.1.
Fill in the details of the proof of Lemma 10.1 by showing that the maximizing point of

$$u - \varphi_\varepsilon$$

converges to (x_0, T) as $\varepsilon \to 0$.

Exercise 10.2.
In the proof of Theorem 10.2, prove carefully that there exists $\mu > 0$ such that

$$t_\varepsilon \geq \mu, \qquad s_\varepsilon \geq \mu$$

for all sufficiently small ε.

Exercise 10.3.
Prove Proposition 10.4 by applying Theorem 10.2 to $u_2 + M$ and u_1.

Exercise 10.4.
Give a detailed proof of Step 3 in Theorem 10.5, namely the spatial Lipschitz estimate obtained from coercivity.

Exercise 10.5.
Explain precisely why the Hamiltonian

$$H(p) = \frac{1}{2}p^2$$

fails to satisfy (1.13), and why uniqueness is still expected in the Lipschitz class.

Exercise 10.6.
Reproduce the proof of finite speed of propagation using the barrier

$$\psi_\varepsilon(x,t) = \xi_\varepsilon(|x| + Ct).$$

14. References

- H. V. Tran, *Hamilton–Jacobi Equations: Theory and Applications*, AMS Graduate Studies in Mathematics 213, 2021.